Monotonicity properties in minimum cost spanning tree problems*

Gustavo Bergantiños
Universidade de Vigo (Spain)

Anirban Kar
Delhi School of Economics (India)

Juan Vidal-Puga
Universidade de Vigo (Spain)

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Abstract

We characterize, in minimum cost spanning tree problems, the family of rules satisfying monotonicity over cost and population. We also prove that the set of allocations induced by the family coincides with the irreducible core.

Keywords: Cost sharing, minimum cost spanning tree problems, monotonicity, irreducible core.

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1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*, for short). A group of agents (denoted by N), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involve some cost. However, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C, where c_{ij} denotes the connection costs between i and j ($i, j \in N \cup \{0\}$).

We assume that agents construct a minimum cost spanning tree (*mcst*). The question is how to divide the cost associated with the *mcst* between the agents. One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help agents/planner to compare different rules and to decide which rule is preferred in a particular situation.

In this paper we focus on two monotonicity properties. Population Monotonicity (PM) claiming that if new agents join a "society" no agent from the "initial society" can be worse off; and Strong Cost Monotonicity (SCM) which claims that if a number of connection costs increase and the rest of the connection costs (if any) remain the same, no agent can be better off[1]. A weaker version of PM is Separability (SEP), which claims that if two groups of agents can connect to the source independently of each other, then we can compute their payments separately.

The main objective of this paper is to study the set of budget-balanced rules satisfying PM and SCM. We focus on two aspects: to characterize the set of rules satisfying PM and SCM and to characterize the set of allocations induced by these rules.

We identify a necessary and sufficient condition for a family of rules to cover all the ones satisfying PM and SCM. In order to describe this condi-

[1]This property is also called *Cost Monotonicity* and *Solidarity* in the literature.

tion, we need to define the so-called *irreducible matrices*, *neighborhoods* and *extra-costs correspondences.*

Given the *mcstp* given by C, Bird (1976) considers the *irreducible matrix* C^*. The irreducible matrix is obtained from C by reducing the cost of the arcs as much as possible, but without reducing the cost of *mcst*. A *neighborhood* is a group of agents that are closer to each other than to any of the other agents or to the source. An *extra-costs correspondence* is a way of dividing any increase in the connection cost between a neighborhood and the source.

The family of rules that satisfy PM and SCM should satisfy a property that says, generally speaking, that the aggregate sum given by the extra-costs correspondence should not decrease when the connection cost between two consecutive neighborhoods is increased.

This property allows us to identify two important subclasses of rules satisfying PM and SCM. These families are the weighted Shapley rules (Bergantiños and Lorenzo-Freire, 2008) and obligation rules (Tijs et al. 2006).

Once we have characterized the rules satisfying PM and SCM, the next step is to study the set of allocations induced by these rules. Bird (1976) associates with each *mcstp* C a cooperative game with transferable utility (N, v_C). We prove that the set of allocations induced by rules satisfying SCM and PM is the core of the game (N, v_{C^*}).

The paper is organized as follows...

2 Notation

Let $U = \{1, 2, 3, ...\}$ be the (infinite) set of possible nodes, and let 0 be special node called the *source.* A *minimum cost spanning tree problem* (*mcstp*) is a pair (N_0, C_0) where $N_0 = N \cup \{0\}$, $N \subset U$ is finite and $C_0 = (c_{ij})_{i,j\in N_0}$ is a matrix with $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N_0$. A *minimum cost connection problem* (*mccp*) is a pair (N, C) where $N \subset U$ is finite and $C = (c_{ij})_{i,j\in N}$ is a matrix with $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N$.

For simplicity, when there is no ambiguity, we write C_0 instead of (N_0, C_0)

and C instead of (N, C).

A *graph* in N_0 is a subset of $\{\{i, j\} : i, j \in N_0, i \neq j\}$. The *cost* of some graph g is defined as $m(g) = \sum_{\{i,j\} \in g} c_{ij}$.

Given $i, j \in N_0$, a *path* between i and j is a graph $\{\{i_{k-1}, i_k\}\}_{k=1}^{K}$ such that $i_0 = i$, $i_K = j$ and $i_k \neq i_{k'}$ whenever $k \neq k'$. A *spanning tree* in N_0 is a graph in N_0 in which there exists exactly one path between any pair of nodes. Let $\mathbb{G}(N)$ (or simply $\mathbb{G}$) denote the set of all graphs in N and let $\mathbb{T}(N)$ (or simply $\mathbb{T}$) denote the set of all spanning trees in N. Analogously for $\mathbb{G}(N_0)$ (or simply $\mathbb{G}_0$) and $\mathbb{T}(N_0)$ (or simply $\mathbb{T}_0$).

A *minimum cost spanning tree* (*mcst*) in C_0 (or in C) is a spanning tree τ in N_0 (or in N) with minimum cost, namely $m(\tau) = \min_{t \in \mathbb{T}_0} m(t)$ (or $m(\tau) = \min_{t \in \mathbb{T}} m(t)$). Since $c_{ij} \geq 0$ for all i, j, it is not difficult to check that $m(\tau) = \min_{g \in \mathbb{G}_0} m(t)$ (or $m(\tau) = \min_{t \in \mathbb{G}} m(t)$).

A *mcst* is not necessarily unique. However, all *mcst* in C_0 (or in C) have the same cost, that we denote as $m(C_0)$ (or $m(C)$).

Given $S \subset N$, we denote as (S, C_S) the restriction of (N, C_S) to S, and we denote as $(S_0, (C_S)_0)$ the restriction of (N_0, C_0) to S.

We denote $\max C := \max_{i,j \in N} c_{ij}$ and $\max C_0 := \max_{i,j \in N_0} c_{ij}$.

Given $i, j \in N$, $\alpha \in \mathbb{R}_+$, we denote as αI_{ij} the matrix C given by $c_{kl} = 0$ for all $\{k, l\} \neq \{i, j\}$ and $c_{ij} = \alpha$.

Let $\mathcal{C}_0$ be the set of all *mcstp* and let $\mathcal{C}$ be the set of all *mccp*.

Given $C_0 \in \mathcal{C}_0$, the *irreducible matrix* of C_0 is defined as C_0^* with

$$c_{ij}^* = \max_{\{k,l\} \in \tau_{ij}} c_{kl}$$

where τ_{ij} is the (unique) path that connects i and j in some *mcst*. This matrix is well-defined, *i.e.* it does not depend on the chosen *mcst*.

Denote $\mathcal{C}_0^* = \{C_0^* : C_0 \in \mathcal{C}_0\}$. Analogously, $\mathcal{C}^* := \{C^* : C \in \mathcal{C}\}$.

A *rule* is a function f that assigns to each $(N_0, C_0) \in \mathcal{C}_0$ a vector $f(N_0, C_0) \in \mathbb{R}^N$, such that $f_i(N_0, C_0)$ (or $f_i(C_0)$ for short), represents the payoff assigned to node $i \in N$. We are interested in rules satisfying the following properties:

Budget Balance (*BB*) $\sum_{i \in N} f_i(N_0, C_0) = m(C_0)$.

Strong Cost Monotonicity (SCM) $C_0 \leq C'_0 \Longrightarrow f(C_0) \leq f(C'_0)$.

Population Monotonicity (PM) $\emptyset \neq S \subset N \Longrightarrow f_i(N_0, C_0) \leq f_i(S_0, (C_S)_0)$ for all $i \in S$.

Separability (SEP) $\emptyset \neq S \subset N, m(N_0, C_0) = m(S_0, (C_S)_0) + m\left((N \backslash S)_0, \left(C_{N\backslash S}\right)_0\right)$ $\Longrightarrow f_i(N_0, C_0) = f_i(S_0, (C_S)_0)$ for all $i \in S$.

It is known (Bergantiños and Vidal-Puga (2007, p. 334)) that PM implies SEP. Moreover, if a rule satisfies SCM, then it only depends on the irreducible matrix, i.e. $f(N_0, C_0) = f(N_0, C_0^*)$. This result follows from Bergantiños and Vidal-Puga (2007, Proposition 3.5).

3 Separability in irreducible matrices

Our first step is to characterize the rules that satisfy SEP and only depend on the irreducible matrix. Notice that all the rules that satisfy PM and SCM belong to this family.

3.1 Neighborhoods

Given $(N_0, C_0) \in \mathcal{C}_0$ and $S \subset N$, $|S| > 1$, we define

$$\delta_S := \min_{i \in S, j \in N_0 \backslash S} c_{ij} - \max_{\{i,j\} \in \tau(S)} c_{ij}$$

where $\tau(S) \in \mathbb{T}(S)$ is a *mcst* in S connecting all the nodes in S. Even though the optimal tree $\tau(S)$ is not necessarily unique, it is not difficult to check that $\max_{\{i,j\} \in \tau(S)} c_{ij}$ does not depend on the particular $\tau(S)$ and hence δ_S is well defined. For $S = \{i\}$, we also define $\delta_{\{i\}} := \min_{j \in N_0 \backslash \{i\}} c_{ij}$.

Roughly speaking, δ_S may be interpreted, when positive, as some kind of "distance" between S and $N_0 \backslash S$. When this is the case, and $|S| > 1$, S is called a neighborhood.

Definition 3.1 *Let (N, C_0) be a mcst problem. We say that $S \subset N$, $|S| > 1$, is a* neighborhood *in C_0 if $\delta_S > 0$. We denote the set of all neighborhoods in C_0 as $Ne(C_0)$.*

Example 3.1 *Let* $N = \{1,2,3,4,5,6\}$ *and* $c_{01} = 50$*,* $c_{12} = 20$*,* $c_{13} = 40$*,* $c_{34} = 10$*,* $c_{15} = 60$*,* $c_{36} = 70$ *and* $c_{ij} > 70$ *otherwise. There are exactly two neighborhoods containing node* 1*:* $\{1,2\}$ *(*$\delta_{\{1,2\}} = 20$*) and* $\{1,2,3,4\}$ *(*$\delta_{\{1,2,3,4\}} = 10$*). Notice that* $\{1,2,3\}$ *is not a neighborhood because* $\delta_{\{1,2,3\}} = 10 - 40 = -30$*.*

Example 3.2 *Let* C_0^* *be the irreducible matrix associated to the matrix presented in the previous example. Hence,* $c_{02}^* = 50$*,* $c_{03}^* = 50$*,* $c_{16}^* = 70$*, and so on. In this new matrix, the neighborhoods are the same as before.*

Notice that, in general, $(C^*)_S \neq (C_S)^*$. Take for example $N = \{1,2,3\}$, $c_{12} = c_{13} = 1$, $c_{23} = 2$ and $S = \{2,3\}$. Then, $c_{23}^* = 1$ and hence $C' = (C^*)_S$ satisfies $c_{23}' = 1$ whereas $C'' = (C_S)^*$ satisfies $c_{23}'' = 2$.

However, the equality holds when S is a neighborhood, as next Proposition shows:

Proposition 3.1 $S \subset N$ *is an neighborhood in* C_0 *if and only if* S *is a neighborhood in* C_0^**. Moreover,* $(C_S)^* = (C^*)_S$ *and*

$$\delta_S = \min_{i \in S, j \in N_0 \setminus S} c_{ij}^* - \max_{i,j \in S} c_{ij}^*.$$

Proof. ($\Longrightarrow$) Assume that S is a neighborhood in C_0. Because of the definition of the irreducible matrix, we have that $\min\limits_{i \in S, j \in N_0 \setminus S} c_{ij} = \min\limits_{i \in S, j \in N_0 \setminus S} c_{ij}^*$. Let $\tau_S \in \mathbb{T}(S)$ be a *mcst* in (S, C_S). Since S is a neighborhood in C_0, τ_S is also an optimal tree in $(S, (C_S)^*)$. Let $C^1 = (C_S)^*$ and let $C^2 = (C^*)_S$. Given $i, j \in S$, let $\tau_{ij} \subset \tau_S$ the (unique) path from i to j. Then,

$$c_{ij}^1 = \max_{\{k,l\} \in \tau_{ij}} c_{kl} = c_{kl}^* = c_{ij}^2$$

and hence$(C_S)^* = (C^*)_S$.

Because of the definition of C^* we have that $\max\limits_{(i,j) \in \tau_S} c_{ij} = \max\limits_{(i,j) \in \tau_S} c_{ij}^* = \max\limits_{(i,j) \in S} c_{ij}^*$. Now,

$$\begin{aligned} \delta_S^* &= \min_{i \in S, j \in N_0 \setminus S} c_{ij}^* - \max_{\{i,j\} \in \tau_S} c_{ij}^* \\ &= \min_{i \in S, j \in N_0 \setminus S} c_{ij} - \max_{\{i,j\} \in \tau_S} c_{ij} = \delta_S \end{aligned}$$

which means that S is an neighborhood in C_0^*.

($\Longleftarrow$) The reciprocal is similar and we omit it. ■

Under Proposition 3.1, for each neighborhood $S \subset N$, we have $(C^*)_S = (C_S)^*$. We denote this matrix as C_S^*.

Proposition 3.2 *If S is a neighborhood in C_0 and $i \in S$, then*

$$S = \left\{ j \in N : c_{ij}^* < \min_{k \in S, l \in N_0 \setminus S} c_{kl}^* \right\}$$

where C_0^* is the irreducible matrix of C_0.

Proof. "$\supset$" Let $j \in N$ be such that $c_{ij}^* < \min_{k \in S, l \in N_0 \setminus S} c_{kl}^*$. If $j \notin S$, then $c_{ij}^* \geq \min_{k \in S, l \in N_0 \setminus S} c_{kl}^*$, which is a contradiction. Hence, $j \in S$.

"$\subset$": Let $j \in N$ be such that $c_{ij}^* \geq \min_{k \in S, l \in N_0 \setminus S} c_{kl}^*$. If $j \in S$, then

$$\delta_S = \min_{k \in S, l \in N_0 \setminus S} c_{kl}^* - \max_{k,l \in S} c_{kl}^* \leq c_{ij}^* - c_{ij}^* = 0$$

which cannot be true because S is a neighborhood. Hence, $j \notin S$. ■

Proposition 3.3 *If S, S' are two neighborhoods in $C_0^* \in \mathcal{C}_0^*$ and $S \cap S' \neq \emptyset$, then either $S \subset S'$ or $S' \subset S$.*

Proof. Let $i \in S \cap S'$. If $\min_{k \in S, l \in N_0 \setminus S} c_{kl}^* \leq \min_{k \in S', l \in N_0 \setminus S'} c_{kl}^*$ then it follows from Proposition 3.2 that $S \subset S'$. If $\min_{k \in S', l \in N_0 \setminus S'} c_{kl}^* \leq \min_{k \in S, l \in N_0 \setminus S} c_{kl}^*$ then it follows from Proposition 3.2 that $S' \subset S$. ■

Corollary 3.1 *For each $i \in N$, there exists a unique family of subsets of N, $S_1, S_2, ..., S_Q$ with $Q \geq 0$ such*[2] *that $\{S_1, ..., S_q\}$ is the set of neighborhoods that contain i, and $S_1 \subset S_2 \subset ... \subset S_q$.*

Proof. It follows from Proposition 3.3. ■

Lemma 3.1 *There exist no neighborhood in C_0 if and only if $\{\{i, 0\}\}_{i \in N}$ is a mcst in C_0.*

[2] Case $q = 0$ covers the situation in which node i has no neighborhoods.

Proof. ($\Longrightarrow$) Assume $\{(i,0)\}_{i\in N}$ is not a *mcst*. Let $\{k,l\}\subset N$ be such that $c_{kl}=\min\limits_{i,j\in N} c_{ij}$. Thus, $c_{kl}<\min\limits_{i\in N} c_{i0}$. Then, $S=\{k\}\cup\left\{i\in N: \max\limits_{\{j,j'\}\in\tau_{ik}} c_{jj'}\leq c_{kl}\right\}$ is a neighborhood in C_0.

($\Longleftarrow$) Assume $\{(i,0)\}_{i\in N}$ is a *mcst*. Then, given any $S\subset N$, we have $\min\limits_{i\in S, j\in N_0\setminus S} c_{ij}=\min\limits_{i\in S} c_{i0}$ and $\max\limits_{\{i,j\}\in\tau(S)} c_{ij}\geq\min\limits_{i\in S} c_{i0}$. Hence

$$\delta_S=\min_{i\in S, j\in N_0\setminus S} c_{ij}-\max_{\{i,j\}\in\tau(S)} c_{ij}\leq 0$$

and S is not a neighborhood. ■

3.2 Extra-costs correspondences

An *extra-costs correspondence* is a function $e:\mathcal{C}^*\times\mathbb{R}_+\to\mathbb{R}_+^U$ satisfying:

- $e_i(C^*,x)=0$ for all $(N,C^*)\in\mathcal{C}^*$, $x\in\mathbb{R}_+$, $i\notin N$, and
- $\sum_{i\in U} e_i(C^*,x)=x$ for all $C^*\in\mathcal{C}^*$, $x\in\mathbb{R}_+$.

Let e be an extra-costs correspondence. We define the rule f^e as follows. Given $(N_0,C_0)\in\mathcal{C}_0$,

$$f_i^e(C_0):=c_{i0}^*-\sum_{\substack{S \text{ neighborhood}\\ S\ni i}}\left(\delta_S-e_i\left(C_S^*,\delta_S\right)\right)$$

for all $i\in N$.

Alternatively,

$$f_i^e(C_0):=c_{i0}^*-\sum_{\substack{S \text{ neighborhood}\\ S\ni i}}\left(\sum_{j\in S\setminus\{i\}} e_j\left(C_S^*,\delta_S\right)\right).$$

Example 3.3 *Let C^* be the matrix presented in example 3.1 and take $i=1$. Hence, $c_{i0}^*=50$ and there are two neighborhoods S with $i\in S$: $S_1=\{1,2\}$ and $S_2=\{1,2,3,4\}$. Moreover, $\delta_{S_1}=20$ and $\delta_{S_2}=10$.*

Let e be defined as $e_j(C^*, x) = \frac{x}{|N|}$ for all $(N, C^*) \in \mathcal{C}$ and $j \in N$ ($e_j(C^*, x) = 0$ otherwise). Then,

$$\begin{aligned} f_1^e(C_0) &= 50 - e_2\left(C^*_{\{1,2\}}, 20\right) \\ &\quad - \left[e_2\left(C^*_{\{1,2,3,4\}}, 10\right) + e_3\left(C^*_{\{1,2,3,4\}}, 10\right) + e_4\left(C^*_{\{1,2,3,4\}}, 10\right)\right] \\ &= 50 - 10 - [2.5 + 2.5 + 2.5] = 32.5. \end{aligned}$$

Proposition 3.4 *Any rule f^e satisfies BB.*

Proof. Let $(N_0, C_0) \in \mathcal{C}_0$. Then,

$$\begin{aligned} \sum_{i \in N} f_i^e(N_0, C_0) &= \sum_{i \in N} c^*_{i0} - \sum_{i \in N} \sum_{\substack{S \text{ neighborhood} \\ S \ni i}} \left(\delta_S - e_i\left(C^*_S, \delta_S\right)\right) \\ &= \sum_{i \in N} c^*_{i0} - \sum_{S \text{ neighborhood}} \left(\sum_{i \in S} \left(\delta_S - e_i\left(C^*_S, \delta_S\right)\right)\right) \\ &= \sum_{i \in N} c^*_{i0} - \sum_{S \text{ neighborhood}} \left(|S| - 1\right) \delta_S. \end{aligned}$$

Thus, it is enough to prove that for each *mcstp* (N_0, C_0),

$$m(C_0) + \sum_{S \text{ neighborhood}} (|S| - 1)\, \delta_S = \sum_{i \in N} c^*_{i0}.$$

Assume first there exists no neighborhood. Under Lemma 3.1, $\{\{i, 0\}\}_{i \in N_0}$ is a *mcst* in (N_0, C_0). Hence, $\{\{i, 0\}\}_{i \in N_0}$ is also a *mcst* in (N_0, C^*_0) and the result is easily checked.

Assume now that there are exactly $k > 0$ neighborhoods and the result is true when there exists less than k neighborhoods. Let S' be a minimal neighborhood (there is no neighborhood S such that $S \subsetneq S'$). Let $\tau_{S'}$ denote a *mcst* in S'. Since S' is minimal, there exists $\alpha \geq 0$ such that $c_{ij} = \alpha$ for all $(i, j) \in \tau_{S'}$.

Let t be a *mcst* in (N_0, C_0). We define C'_0 as $c'_{ij} = \alpha + \delta_{S'}$ if $\{i, j\} \subset S'$ and $c'_{ij} = c_{ij}$ otherwise. It is not difficult to check that:

- t is also a *mcst* in (N_0, C'_0);

- $c'^{*}_{i0} = c^{*}_{i0}$ for all $i \in N$;
- $m(C'_0) = m(C'_0) + (|S'| - 1)\,\delta_{S'}$; and
- $\{S : S \text{ is a neighborhood in } C'_0\} = \{S : S \text{ is a neighborhood in } C_0\} \setminus \{S'\}$.

Now, applying the induction hypothesis, we have

$$\begin{aligned} & m(C_0) + \sum_{S \text{ neighborhood in } C_0} (|S| - 1)\,\delta_S \\ = \; & m(C'_0) - (|S'| - 1)\,\delta_{S'} + \sum_{S \text{ neighborhood in } C_0} (|S| - 1)\,\delta_S \\ = \; & m(C'_0) + \sum_{S \text{ neighborhood in } C'_0} (|S| - 1)\,\delta_S \\ = \; & \sum_{i \in N} c'^{*}_{i0} = \sum_{i \in N} c^{*}_{i0}. \end{aligned}$$

■

Theorem 3.1 *The rules f^e are the only ones that satisfy BB, SEP and only depend on the irreducible matrix.*

Proof. We have just proved that f^e satisfies BB. Moreover, it is obvious that it only depends on the irreducible matrix. In order to prove SEP, let $S \subset N$ such that $m(N_0, C_0) = m(S_0, C_0) + m((N \backslash S)_0, C_0)$. Given $i \in S$, it is straightforward to check that $Ne(N_0, C_0) = Ne(S_0, C_0) \cup Ne((N \backslash S)_0, C_0)$. Hence, $f_i^e(N_0, C_0) = f_i^e(S_0, C_0)$ and this proves that f is separable.

We now prove that if f satisfies BB, SEP and $f(C_0) = f(C_0^*)$, then $f = f^e$ for some extra-costs correspondence e. Let f be such a rule.

Given $(N, C^*) \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$, we define $\left(N_0, C_0^{*(a)}\right) \in \mathcal{C}_0$ as the *mcstp* given by $c_{ij}^{*(a)} = c_{ij}^*$ for all $i, j \in N$ and $c_{i0}^{*(a)} = a$ for all $i \in N$. It is straightforward to check that $C_0^{*(a)} \in \mathcal{C}_0^*$ when $a \geq \max C^*$.

For all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$ we define

$$e_i(C^*, x) = f_i\left(C_0^{*(\max C^* + x)}\right) - f_i\left(C_0^{*(\max C^*)}\right).$$

Given $i \notin N$ we define $e_i(C^*, x) = 0$.

We first prove that e is an extra-costs correspondence.

- By definition, $e_i(C^*, x) = 0$ for all $(N, C^*) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, $i \notin N$.
- Besides,

$$\begin{aligned}
\sum_{i \in U} e_i(C^*, x) &= \sum_{i \in N} e_i(C^*, x) \\
&= m\left(C_0^{*(\max C^* + x)}\right) - m\left(C_0^{*(\max C^*)}\right) \\
&= m(C^*) + \max C^* + x - m(C^*) - \max C^* \\
&= x.
\end{aligned}$$

Hence, e is an extra-costs correspondence.

We need to prove that $f = f^e$. We proceed by induction on the number of neighborhoods $Ne(C_0)$. Assume $|Ne(C_0)| = 0$.

Under Lemma 3.1, $\{(i,0)\}_{i \in N}$ is a *mcst* in C_0. Since f satisfies SEP, $f_i(C_0) = f_i(\{i\}_0, C_0)$. Under BB, $f_i(C_0) = c_{i0}$. Moreover, since $\{(i,0)\}_{i \in N}$ is a *mcst* in C_0, we have $c_{i0} = c^*_{i0}$ for all $i \in N$ and hence $f^e(C_0) = f(C_0)$.

Assume now the result is true for *mcstp* with less than $|Ne(C_0)|$ neighborhoods.

Assume first that $\max C^* \geq \max\limits_{i \in N} c^*_{i0}$. It is not difficult to check that N is separable, namely, there exists $S \subset N$, $S \neq \varnothing$, and $S \neq N$ such that $m(N_0, C_0) = m(S_0, C_0) + m((N \backslash S)_0, C_0)$. Under SEP, $f_i(N_0, C_0) = f_i(S_0, C_0)$ for all $i \in S$ and $f_i(N_0, C_0) = f_i((N \backslash S)_0, C_0)$ for all $i \in N \backslash S$. Repeating this argument we can find a partition $\{S_1, ..., S_p\}$ of N satisfying that for each $k = 1, ...p$ $\max C^*_{S_k} < \max\limits_{i \in S_k} c^*_{i0}$ and $f_i(N_0, C_0) = f_i((S_k)_0, C_0)$ for each $i \in S_k$.

Hence, we can assume that $\max C^* < \max\limits_{i \in N} c^*_{i0}$. Since C^* is irreducible, $\max\limits_{i \in N} c^*_{i0} = c^*_{i0}$ for all $i \in N$. Hence, $N \in Ne(C_0)$ and $\delta_N = \max\limits_{i \in N} c^*_{i0} - \max C^*$. Since f only depends on the irreducible matrix, $f(C_0) = f(C^*_0)$. Now, given $i \in N$,

$$\begin{aligned}
f_i(C_0) &= f_i(C^*_0) = f_i\left(C_0^{*(\max C^* + \delta_N)}\right) \\
&= e_i(C^*, \delta_N) + f_i\left(C_0^{*(\max C^*)}\right).
\end{aligned}$$

Let $C_0' = C_0^{*(\max C^*)}$. It is straightforward to check that C_0' is irreducible. Moreover, $Ne\,(C_0^*) = Ne\,(C_0') \cup \{N\}$. For each $S \in Ne\,(C_0')$, $\delta_S = \delta_S'$, and $c_{i0}'^* = c_{i0}^* - \delta_N$. Hence, applying the induction hypothesis, for each $i \in N$,

$$\begin{aligned} f_i\,(C_0) &= e_i\,(C^*, \delta_N) + f_i\,(C_0') \\ &= e_i\,(C^*, \delta_N) + c_{i0}'^* + \sum_{S \in Ne\left(C_0'\right)} \left(e_i\,(C_S^*, \delta_S) - \delta_S\right) \\ &= e_i\,(C^*, \delta_N) + c_{i0}^* - \delta_N + \sum_{S \in Ne\left(C_0'\right)} \left(e_i\,(C_S^*, \delta_S) - \delta_S\right) \\ &= c_{i0}^* + \sum_{S \in Ne\left(C_0^*\right)} \left(e_i\,(C_S^*, \delta_S) - \delta_S\right) \\ &= f_i^e\,(C_0)\,. \end{aligned}$$

■

4 The main characterization

Given $(N^1, C^1), (N^2, C^2) \in \mathcal{C}$, $N^1 \cap N^2 = \emptyset$, and $a \in \mathbb{R}_+$, we define

$$\left(N^1 \cup N^2, C^1 \oplus_a C^2\right)$$

as the *mccp* C given by $c_{ij} = c_{ij}^\alpha$ if $i, j \in N^\alpha$ for some $\alpha \in \{1, 2\}$, and

$$c_{ij} = a + \max C^1$$

for all $i \in N^1$, $j \in N^2$.

For convenience, we write $C^1 \oplus_a C^2 \oplus_b C^3$ instead of $(C^1 \oplus_a C^2) \oplus_b C^3$, and so on.

Given $a = (a_1, ..., a_\Gamma)$, $\left(C^1, ..., C^\Gamma\right)$, and $\gamma \leq \Gamma$ we denote

$$C^\gamma\,(a) = C^1 \oplus_{a_1} C^2 \oplus_{a_2} ... \oplus_{a_{\gamma-1}} C^\gamma.$$

Notice that, given $\gamma > 1$,

$$C^\gamma\,(a) = C^{\gamma-1}\,(a) \oplus_{a_{\gamma-1}} C^\gamma. \tag{1}$$

Proposition 4.1 *(i) Given $(N', C'), (N'', C'') \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$ with $N' \cap N'' = \emptyset$ and $a \geq \max C'' - \max C'$, then $C' \oplus_a C'' \in \mathcal{C}^*$.*

(ii) Given a disjoint sequence $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $\Gamma > 1$, $a \in \mathbb{R}_+^\Gamma$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$, then $C^\gamma(a) \in \mathcal{C}^*$ and $C^\gamma(a') \in \mathcal{C}^*$ for all $\gamma = 1, ..., \Gamma$, where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$.

Proof. (i) Let $C = C' \oplus_a C''$. It is easily checked that $a + \max C' = \max C$. Hence, we can find a *mcst* t in C and C^* such that $t = t^1 \cup t^2 \cup \{(k^1, k^2)\}$ where t^1 is *mcst* in C', t^2 is a *mcst* in C'', $k^1 \in N^1$ and $k^2 \in N^2$. Since $c_{k^1k^2} = \max C \geq c_{ij}$ for all $(i,j) \in t^1 \cup t^2$ we can deduce, using the definition of irreducible matrix, that $C = C^*$.

(ii) We assume $\gamma > 1$, since the case $\gamma = 1$ is trivial. We proceed by induction on Γ. For $\Gamma = 2$, the result follows from (i) because $a'_1 = a_1 + y \geq a_1 \geq \max C^2 - \max C^1$. Assume the result is true for sequences with less than Γ *mcstp*'s, $\Gamma \geq 3$. Under the induction hypothesis, we have $C^\gamma(b)$, $C^\gamma(b') \in \mathcal{C}^*$ where $\gamma = 1, ..., \Gamma - 1$, $b = (a_1, ..., a_{\Gamma-1})$ and $b' = (a_1 + y, a_2 - y, a_3, ..., a_{\Gamma-1})$. Now, it is clear that $C^\gamma(a) = C^\gamma(b)$ and $C^\gamma(a') = C^\gamma(b')$ for all $\gamma = 1, ..., \Gamma - 1$. Hence, the result holds for any $\gamma < \Gamma$. Assume now $\gamma = \Gamma$. We have

$$C^\Gamma(a) \stackrel{(1)}{=} C^{\Gamma-1}(a) \oplus_{a_{\Gamma-1}} C^\Gamma(a) \stackrel{(i)}{\in} \mathcal{C}^*$$

and

$$C^\Gamma(a') \stackrel{(1)}{=} C^{\Gamma-1}(a') \oplus_{a'_{\Gamma-1}} C^\Gamma(a').$$

In order to apply (i) to this last expression (so that $C^\Gamma(a') \in \mathcal{C}^*$) we have to prove that

$$a'_{\Gamma-1} \geq \max C^\Gamma(a') - \max C^{\Gamma-1}(a') \tag{2}$$

It is straightforward to check that $\max C^\gamma(a') = \max C^\gamma(a)$ for all $\gamma \neq 2$, whereas $\max C^2(a') = \max C^2(a) + y$. Hence, for $\Gamma > 3$,

$$\max C^\Gamma(a') - \max C^{\Gamma-1}(a') = \max C^\Gamma(a) - \max C^{\Gamma-1}(a) \leq a_{\Gamma-1} = a'_{\Gamma-1}$$

and for $\Gamma = 3$,

$$\max C^3(a') - \max C^2(a') = \max C^3(a) - \max C^2(a) - y \leq a_2 - y = a'_2.$$

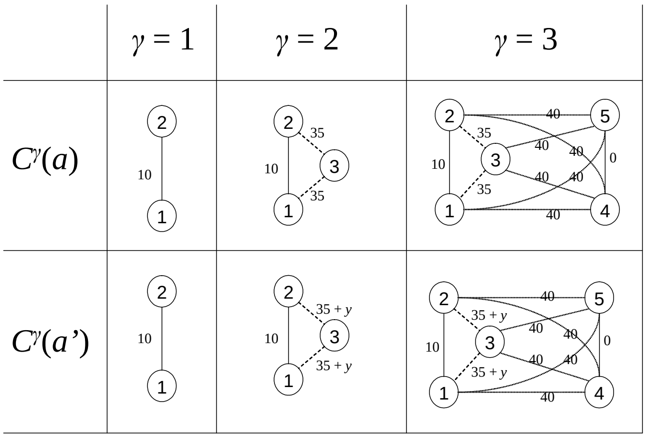

Figure 1: Minimum cost connection problems $C^{\gamma}(a), C^{\gamma}(a')$ for $\gamma = 1, 2, 3$. PRO requires the aggregate assignment of extra costs for players 1, 2, 4 and 5 to be not higher with a than with a'.

■

Definition 4.1 *We say that an extra-costs correspondences* e *satisfies the property of* Proximity *(PRO) if for all disjoint sequences* $\{(N^{\gamma}, C^{\gamma})\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $\Gamma \geq 1$, $i \in N^{\gamma_i}$ *with* $\gamma_i \neq 2$, $a \in \mathbb{R}_+^{\Gamma}$ *with* $a_{\gamma} \geq \max C^{\gamma+1} - \max C^{\gamma}$ *for all* $\gamma = 1, ..., \Gamma - 1$*, and* $y \in [0, a_2]$ *(*$y \geq 0$ *when* $\Gamma = 1$*), we have*

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a'), a'_{\gamma}\right) \geq \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a), a_{\gamma}\right)$$

where $a' = (a_1 + y, a_2 - y, a_3, ..., a_{\Gamma})$ *(*$a' = (a_1 + y)$ *when* $\Gamma = 1$*).*

Example 4.1 *Let* $\Gamma = 3$, $N^1 = \{1, 2\}, c_{12}^1 = 10, N^2 = \{3\}, N^3 = \{4, 5\}$ *and* $c_{45}^3 = 0$*. Then,* $a = (25, 15, 20)$ *and* $a' = (25 + y, 15 - y, 20)$ *with* $y \in [0, 15]$ *satisfy the conditions imposed on the definition of PRO:* $a_1 = 25 \geq 0 - 10 = \max C^2 - \max C^1$, $a_2 = 15 \geq 0 - 0 = \max C^3 - \max C^2$*.* $C^{\gamma}(a)$ *and* $C^{\gamma}(a')$ *are described in Figure 1.*

Let $\widehat{P}$ be the set of all rules f^e such that e satisfies PRO.

Theorem 4.1 *Let f be a rule. Then, f satisfies BB, SCM and PM if and only if $f \in \widehat{P}$.*

Proof. We already now (by Proposition 3.4) that any $f \in \widehat{P}$ satisfies BB.

We now prove that if $f \in \widehat{P}$, then f satisfies SCM and PM. Assume that $f = f^e$ where e satisfies PRO.

Let $\mathcal{C}_0^N$ denote the subset of *mcstp* whose set of nodes is N.

Following Tijs, Moretti, Branzei and Norde (2004), we define the set Σ_{N_0} of linear orders on the arcs of C_0 as the set of all bijections $\sigma : \left\{1, ..., \binom{n+1}{n}\right\} \to \{\{i,j\} : i,j \in N_0\}$. For each *mcstp* (N_0, C_0), there exists at least one linear order $\sigma \in \Sigma_{N_0}$ such that $c_{\sigma(1)} \leq c_{\sigma(2)} \leq ... \leq c_{\sigma\left(\binom{n+1}{n}\right)}$. For any $\sigma \in \Sigma_{N_0}$, we define the set

$$K^\sigma = \left\{C_0 \in \mathcal{C}_0^N : c_{\sigma(k)} \leq c_{\sigma(k+1)} \text{ for all } k = 1, 2, ...\right\},$$

which we call the *Kruskal cone* with respect to σ. One can easily see that $\bigcup_{\sigma \in \Sigma_{N_0}} K^\sigma = \mathcal{C}_0^N$.

We say that a nonempty set $S \subset N$ is a *quasi-neighborhood* in C_0 if $\delta_S \geq 0$. Let $qNe(C_0) = \{S \subset N, S \neq \emptyset : \delta_S \geq 0\}$ denote the set of quasi-neighborhoods in C_0. Clearly, $Ne(C_0) \subseteq qNe(C_0)$.

We now prove that f satisfies SCM. It is enough to prove that $f(N_0, C_0) \leq f(N_0, C_0')$ when there exists $\{k,l\} \subset N_0$ such that $c'_{kl} > c_{kl}$ and $c'_{ij} = c_{ij}$ otherwise. Let (k,l), C_0 and C_0' be defined in this way.

For any $t \in [0,1]$, the *mcstp* (N_0, C_0^t) defined as $c_{ij}^t = (1-t)c_{ij} + tc'_{ij}$ satisfies $c'_{kl} \geq c_{kl}^t \geq c_{kl}$ and $c_{ij}^t = c_{ij}$ otherwise. Since Σ_{N_0} is a finite set, there exist a sequence $\{t^1, t^2, ...t^p\} \subset [0,1]$ with $t^1 = 0$ and $t^p = 1$ such that, for all r, we have $t^r < t^{r+1}$ and C^{t^r} and $C^{t^{r+1}}$ belong to the same Kruskal cone.

Hence, it is enough to prove that $f(N_0, C_0) \leq f(N_0, C_0')$ when both C_0 and C_0' belong to the same Kruskal cone. An immediate consequence is that there exists a common *mcst* t in both C_0 and C_0'.

By definition of f, it is obvious that $f(N_0, C_0) = f(N_0, C_0^*)$. Hence, if $\{k,l\} \notin t$, then $C_0^* = C_0'^*$ and thus

$$f(N_0, C_0) = f(N_0, C_0^*) = f(N_0, C_0'^*) = f(N_0, C_0').$$

Hence, we assume $\{k,l\} \in t$. This implies $c_{kl} = c^*_{kl}$ and $c'_{kl} = c'^*_{kl}$. Let $\alpha = c'^*_{kl} - c^*_{kl} > 0$.

Another consequence of C_0, C'_0 being in the same Kruskal cone is that, for any $S \subset N$, $|S| > 1$, there exist $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \backslash S$ with $\{i^2, j^2\} \in \tau(S)$ such that

$$\begin{aligned}\delta_S &= \min_{i' \in S, j' \in N_0 \backslash S} c_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c_{i'j'} = c_{i^1j^1} - c_{i^2j^2} \text{ and} \\ \delta'_S &= \min_{i' \in S, j' \in N_0 \backslash S} c'_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c'_{i'j'} = c'_{i^1j^1} - c'_{i^2j^2}.\end{aligned}$$

Thus δ_S and δ'_S cannot have opposite sign. Namely, $\delta_S > 0$ implies $\delta'_S \geq 0$.

From this, it is straightforward to check that $Ne(C_0) \subset qNe(C'_0)$ and, analogously, $Ne(C'_0) \subset qNe(C_0)$.

Given any $X \subset 2^N$ with $Ne(C_0) \subseteq X \subseteq qNe(C_0)$, we have

$$f_i(N_0, C_0) = c^*_{i0} - \sum_{S \in X : S \ni i} (\delta_S - e_i(C^*_S, \delta_S)) \tag{3}$$

for all $i \in N$. The reason is that for any $S \in qNe(C_0) \backslash Ne(C_0)$, $\delta_S = 0$ and hence $\delta_S - e_i(C^*_S, \delta_S) = 0 - e_i(C^*_S, 0) = 0$.

We define $X = Ne(C_0) \cup Ne(C'_0)$. Clearly, $Ne(C_0) \subseteq X \subseteq qNe(C_0)$ and $Ne(C'_0) \subseteq X \subseteq qNe(C'_0)$.

Fix $i \in N$. We need to prove that $f_i(N_0, C_0) \leq f_i(N_0, C'_0)$. Under (3), we have

$$\begin{aligned}f_i(N_0, C_0) &= c^*_{i0} - \sum_{S \in X : S \ni i} (\delta_S - e_i(C^*_S, \delta_S)) \\ f_i(N_0, C'_0) &= c'^*_{i0} - \sum_{S \in X : S \ni i} (\delta'_S - e_i(C'^*_S, \delta'_S)).\end{aligned}$$

We have seen above that

$$\delta_S = c_{i^1j^1} - c_{i^2j^2} \text{ and } \delta'_S = c'_{i^1j^1} - c'_{i^2j^2}$$

for some $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \backslash S$ with $\{i^2, j^2\} \in t_S$.

By hypothesis, $c_{jj'} = c'_{jj'}$ for all $\{j, j'\} \neq \{k, l\}$. Hence, $\delta_S = \delta'_S$ unless $\{i^1, j^1\} = \{k, l\}$ or $\{i^2, j^2\} = \{k, l\}$.

Given $S \in X$ and $\delta_S \neq \delta'_S$ we study both cases:

1. If $\{i^1, j^1\} = \{k, l\}$, then $\delta'_S = \delta_S + \alpha$. Moreover, there can be at most two such S. One of them contains node k (if any) and the other contains node l (if any). Assume, on the contrary, that there exist two $S' \in X, S \neq S'$ with $k \in S \cap S'$ (the case for $l \in S$ is analogous). Hence,

$$c'_{kl} = c'^*_{kl} = \min_{i' \in S, j' \in N_0 \setminus S} c'^*_{i'j'} = \min_{i' \in S', j' \in N_0 \setminus S'} c'^*_{i'j'}.$$

Since $k \in S \cap S'$, under Corollary 3.1, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

$$\begin{aligned} c'^*_{kl} &= \min_{i' \in S, j' \in N_0 \setminus S} c'^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c'^*_{i'j'} \\ &\leq \max_{i', j' \in S'} c'^*_{i'j'} \leq \min_{i' \in S', j' \in N_0 \setminus S'} c'^*_{i'j'} = c'^*_{kl} \end{aligned}$$

which implies that no inequality is strict. In particular, $\max_{i', j' \in S'} c'^*_{i'j'} = c'^*_{kl}$. Since $\{k, l\} \nsubseteq S'$, $\max_{i', j' \in S'} c'^*_{i'j'} = \max_{i', j' \in S'} c^*_{i'j'}$ and hence

$$\delta_{S'} = \min_{i' \in S', j' \in N_0 \setminus S'} c^*_{i'j'} - \max_{i', j' \in S'} c^*_{i'j'} = c^*_{kl} - c'^*_{kl} = -\alpha < 0,$$

which is a contradiction.

2. If $\{i^2, j^2\} = \{k, l\}$, then $\delta'_S = \delta_S - \alpha$. Moreover, there can be at most one such S. Assume, on the contrary, that there exists $S' \in X$, $S \neq S'$, $k, l \in S \cap S'$, and

$$c_{kl} = c^*_{kl} = \max_{i', j' \in S} c^*_{i'j'} = \max_{i', j' \in S'} c^*_{i'j'}.$$

Since $k \in S \cap S'$, under Corollary 3.1, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

$$c^*_{kl} = \max_{i', j' \in S} c^*_{i'j'} \leq \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c^*_{i'j'} \leq \max_{i', j' \in S'} c^*_{i'j'} = c^*_{kl}$$

which implies that no inequality is strict. Thus, $\min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} = c^*_{kl}$ and hence

$$\delta_S = \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} - \max_{i', j' \in S} c^*_{i'j'} = c^*_{kl} - c^*_{kl} = 0,$$

which implies $\delta'_S = \delta_S - \alpha = -\alpha < 0$, which is a contradiction.

Let $S^k = \{j \in N_0 : c'^*_{kj} < c'^*_{kl}\}$ and let $S^l = \{j \in N_0 : c'^*_{kj} < c'^*_{kl}\}$. Both S^k and S^l are nonempty (because $k \in S^k$ and $l \in S^l$) and disjoint (it follows from $\{k,l\} \in t$). Since they are disjoint, we can assume w.l.o.g. $0 \notin S^k$. Let $S_1 = S^k$. If $|S_1| > 1$, then

$$\begin{aligned} l &\notin S_1, \\ c'^*_{kl} &= \min_{i' \in S_1, j' \in N_0 \setminus S_1} c'^*_{i'j'}, \\ \delta'_{S_1} &= c'^*_{kl} - \max_{i',j' \in S} c'^*_{i'j'} > 0 \end{aligned}$$

and hence either $S_1 \in Ne\,(C'_0)$ or $S_1 = \{k\}$.

Assume that $S_1 \in Ne\,(C'_0)$. Since C_0 and C'_0 are in the same Kruskal cone, $\delta_{S_1} = c^*_{i^1j^1} - c^*_{i^2j^2}$ and $\delta'_{S_1} = c'^*_{i^1j^1} - c'^*_{i^2j^2}$. Since $\delta'_{S_1} > 0$ we deduce that $\delta_{S_1} \geq 0$. Hence $S_1 \in qNe\,(C_0)$. Now, it is not difficult to check that S_1 satisfies condition 1, hence $\delta'_{S_1} = \delta_{S_1} + \alpha$ when $|S_1| > 1$.

Let $S_2 = \{j \in N_0 : c^*_{kj} \leq c^*_{kl}\}$. Clearly, $\{k,l\} \subset S_2$. Notice that if $0 \in S_2$ then $S_2 \notin X$. It is straightforward to check that if $0 \notin S_2$ then $S_2 \in X$. Besides $S_1 \subsetneq S_2$ and there is no $S \in X$, $S \neq S_1$, such that $S_1 \subsetneq S \subsetneq S_2$.

In case $0 \notin S_2$, it is not difficult to check that S_2 satisfies condition 2, hence $\delta'_{S_2} = \delta_{S_2} - \alpha$.

Let $F = \{S \in Ne\,(C_0) : S_1 \subset S, \delta_S = \delta'_S\}$ and let $F' = \{S \in Ne\,(C'_0) : S_1 \subset S, \delta_S = \delta'_S\}$. It is not difficult to check that $F = F'$ ($F = F' = \emptyset$ is also possible) and, moreover, $S_1, S_2 \notin F$. By Proposition 3.3 we can assume $F = \{S_3, S_4, ..., S_\Gamma\}$ for some $\Gamma \geq 2$ ($\Gamma = 2$ when $F = \emptyset$) and $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 3, ..., \Gamma - 1$.

Let $G = \{S \in X : S_1 \subset S\}$. Clearly, either $G = \{S_1, ..., S_\Gamma\}$ (when $S_1 \in Ne\,(C'_0)$) or $G = \{S_2, ..., S_\Gamma\}$ (when $S_1 = \{k\}$). Moreover, $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 1, 2, ..., \Gamma - 1$.

If $i \notin S_\Gamma$, it is straightforward to check that $f_i\,(N_0, C_0) = f_i\,(N_0, C'_0)$.

Hence, we assume $i \in S_\gamma$ for some $\gamma \in \{1, ..., \Gamma\}$. Let γ_i be the minimum of these γ's.

We have two cases:

Case 1: $\Gamma = 1$. This means $S_2 \notin X$. Since $\delta_{S_2} \geq 0$, we have $0 \in S_2$, which implies $c^*_{k0} \leq c^*_{kl}$ and also $c'^*_{k0} \leq c'^*_{kl}$.

Subcase 1.1: $S_1 = \{k\} = \{i\}$. This implies $X = \emptyset$ and hence

$$f_i\left(N_0, C_0'\right) - f_i\left(N_0, C_0\right) = c_{i0}'^* - c_{i0}^* \geq 0.$$

Subcase 1.2: $S_1 \in X$. This implies $c_{k0}'^* \geq c_{kl}'^*$ and hence $c_{k0}'^* = c_{kl}'^*$. This implies $c_{i0}'^* - c_{i0}^* = \alpha$. Moreover, $C_{S_1}^* = C_{S_1}'^*$. Hence,

$$\begin{aligned}
& f_i\left(N_0, C_0'\right) - f_i\left(N_0, C_0\right) \\
= \;& c_{i0}'^* - \left(\delta_{S_1}' - e_i\left(C_{S_1}'^*, \delta_{S_1}'\right)\right) - c_{i0}^* + \left(\delta_{S_1} - e_i\left(C_{S_1}^*, \delta_{S_1}\right)\right) \\
= \;& c_{i0}'^* - c_{i0}^* - \left(\delta_{S_1} + \alpha - e_i\left(C_{S_1}^*, \delta_{S_1} + \alpha\right)\right) + \left(\delta_{S_1} - e_i\left(C_{S_1}^*, \delta_{S_1}\right)\right) \\
= \;& e_i\left(C_{S_1}^*, \delta_{S_1} + \alpha\right) - e_i\left(C_{S_1}^*, \delta_{S_1}\right) \geq 0
\end{aligned}$$

where the last inequality comes from applying PRO to $\left\{\left(S_1, C_{S_1}^*\right)\right\}$ with $\Gamma = 1$, $a_1 = \delta_{S_1}$ and $y = \alpha$.

Case 2: $\Gamma > 1$. This means $S_2 \in X$ and hence $0 \notin S^l$. Thus we can take $S_1 = S^k$ or $S_1 = S^l$. It is not difficult to check that $S_2 = S^k \cup S^l$. If $i \in S_2$ we choose S_1 such that $i \in S_1$. Thus, $\gamma_i \neq 2$. This implies $c_{i0}'^* = c_{i0}^*$.

In this case,

$$\begin{aligned}
& f_i\left(N_0, C_0'\right) - f_i\left(N_0, C_0\right) \\
= \;& c_{i0}'^* - c_{i0}^* - \sum_{S \in X : S \ni i} \left(\delta_S' - \delta_S - e_i\left(C_S'^*, \delta_S'\right) + e_i\left(C_S^*, \delta_S\right)\right).
\end{aligned}$$

For any $S \in X \backslash G$ with $i \in S$, we have $C_S^* = C_S'^*$, which also implies $\delta_S = \delta_S'$. Hence,

$$\begin{aligned}
& f_i\left(N_0, C_0'\right) - f_i\left(N_0, C_0\right) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} \left(-\delta_{S_\gamma}' + \delta_{S_\gamma} + e_i\left(C_{S_\gamma}'^*, \delta_{S_\gamma}'\right) - e_i\left(C_{S_\gamma}^*, \delta_{S_\gamma}\right)\right) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C_{S_\gamma}'^*, \delta_{S_\gamma}'\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C_{S_\gamma}^*, \delta_{S_\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(\delta_{s_\gamma}' - \delta_{s_\gamma}\right).
\end{aligned}$$

The last term is zero, because $\delta_{S_1}' = \delta_{S_1} + \alpha$, $\delta_{S_2}' = \delta_{S_2} - \alpha$, and $\delta_{S_\gamma}' = \delta_{S_\gamma}$ otherwise (remark that $\gamma_i \neq 2$). Hence,

$$f_i\left(N_0, C_0'\right) - f_i\left(N_0, C_0\right) = \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i\left(C_{S_\gamma}'^*, \delta_{S_\gamma}'\right)\right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i\left(C_{S_\gamma}^*, \delta_{S_\gamma}\right)\right).$$

We now define $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma}$, $a \in \mathbb{R}_+^\Gamma$ and $y \in [0, a_2]$ so that $e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) = e_i\left(C^\gamma(a'), a'_\gamma\right)$ and $e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right) = e_i\left(C^\gamma(a), a_\gamma\right)$ for all γ. Under PRO, this will prove that the above expression is nonnegative.

Let $N^1 = S_1$, $C^1 = C^*_{N^1}$, and $a_1 = \delta_{S_1}$. In general, for any $\gamma = 2, ..., \Gamma$, $N^\gamma = S_\gamma \backslash S_{\gamma-1}$, $C^\gamma = (C^*)_{N^\gamma}$, and $a_\gamma = \delta_{S_\gamma}$. We also define $y = \alpha$. Since $c'^*_{kl} = c^*_{kl} + \alpha$, it is straightforward to check that $\alpha \leq a_2$ and hence $y \in [0, a_2]$.

Clearly, $C'^*_{S_1} = C^1$. Now, we prove that $C'^*_{S_2} = C^1 \oplus_{a_1+\alpha} C^2 = C^2(a')$. Let $C^\alpha = C'^*_{S_2}$ and $C^\beta = C^1 \oplus_{a_1+\alpha} C^2$. Clearly, $C^\alpha = (C_{S_2} + \alpha I_{kl})^*$.

It is straightforward to check that $c^\alpha_{ij} = c^\beta_{ij}$ for all $i, j \in N^1$ and all $i, j \in N^2$. Let $k^1 \in N^1$ and $k^2 \in N^2$. Then,

$$\begin{aligned} c^\beta_{k^1k^2} &= \max C^1 + a_1 + \alpha = \max C^1 + \delta_{S_1} + \alpha = \min_{\substack{i \in N^1 \\ j \in N_0 \backslash N^1}} c_{ij} + \alpha \\ &= c_{kl} + \alpha = c^\alpha_{k^1k^2}. \end{aligned}$$

Analogously, $C'^*_{S_3} = (C_{S_3} + \alpha I_{kl})^* = (C^1 \oplus_{a_1+\alpha} C^2) \oplus_{a_2-\alpha} C^3 = C^3(a')$. In general, $C'^*_{S_\gamma} = \left(C_{S_\gamma} + \alpha I_{kl}\right)^* = C^1 \oplus_{a_1+\alpha} C^2 \oplus_{a_1-\alpha} C^3 \oplus_{a_3} ... \oplus_{a_{\gamma-1}} C^\gamma = C^\gamma(a')$ for all $\gamma = 3, ..., \Gamma$.

Similarly, we can prove that $C^*_{S_\gamma} = C^\gamma(a)$ for all $\gamma = 1, ..., \Gamma$.

Hence, by applying PRO, we have

$$f_i(N_0, C'_0) - f_i(N_0, C_0) \geq 0.$$

We now prove that f satisfies PM. By Theorem 3.1, we know that f satisfies SEP. We must prove that for each $mcstp$ (N_0, C_0) and $j \in N$, $f_i(N_0, C_0) \leq f_i((N \backslash \{j\})_0, C_0)$ for all $i \in N \backslash \{j\}$. Let (N_0, C'_0) be defined as $c'_{ii'} = c_{ii'}$ for all $i, i' \in N \backslash \{j\}$ and $c'_{ij} = \max C_{N_0 \backslash \{j\}}$ for all $i \in N_0 \backslash \{j\}$. Clearly, $m(N_0, C'_0) = m((N \backslash \{j\})_0, C'_0) + m(\{j\}_0, C'_0)$ and hence, under SEP, $f_i(N_0, C'_0) = f_i((N \backslash \{j\})_0, C'_0)$ for all $i \in N \backslash \{j\}$. Given $i \in N \backslash \{j\}$, under SCM,

$$f_i(N_0, C_0) \leq f_i(N_0, C'_0) = f_i((N \backslash \{j\})_0, C'_0) = f_i((N \backslash \{j\})_0, C_0).$$

We now prove that if f satisfies BB, SCM and PM, then $f \in \widehat{P}$.

We define e as in the proof of Theorem 3.1. Namely, for all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$,

$$e_i\left(C^*, x\right) = f_i\left(C_0^{*(\max C^* + x)}\right) - f_i\left(C_0^{*(\max C^*)}\right).$$

and $e_i\left(C^*, x\right) = 0$ for all $i \notin N$. We already proved (proof of Theorem 3.1) that e is an extra-cost correspondence and $f = f^e$.

Hence, we only need to check that e satisfies PRO. Let $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset C^*$ be a disjoint sequence with $\Gamma \geq 1$, $i \in N^{\gamma_i}$ with $\gamma_i \neq 2$, $a \in \mathbb{R}_+^\Gamma$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$ and $y \in [0, a_2]$ (or simply $y \geq 0$, when $\Gamma = 1$).

Assume first $\Gamma = 1$. We need to prove

$$e_i\left(C^1, a_1 + y\right) - e_i\left(C^1, a_1\right) \geq 0.$$

Let $C = C^1$. By definition,

$$\begin{aligned}
& e_i\left(C, a_1 + y\right) - e_i\left(C, a_1\right) \\
= \; & f_i\left(C_0^{*(\max C^* + a_1 + y)}\right) - f_i\left(C_0^{*(\max C^*)}\right) - f_i\left(C_0^{*(\max C^* + a_1)}\right) + f_i\left(C_0^{*(\max C^*)}\right) \\
= \; & f_i\left(C_0^{*(\max C^* + a_1 + y)}\right) - f_i\left(C_0^{*(\max C^* + a_1)}\right) \geq 0
\end{aligned}$$

where the last inequality comes from the fact that $C_0^{*(\max C^* + a_1 + y)} \geq C_0^{*(\max C^* + a_1)}$ and f satisfy SCM.

Assume now $\Gamma > 1$. We need to prove

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma\left(a'\right), a'_\gamma\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma\left(a\right), a_\gamma\right) \geq 0$$

where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$ and $C^\gamma(b) = C^1 \oplus_{b_1} C^2 \oplus_{b_2} ... \oplus_{b_{\gamma-1}} C^\gamma$ for all $\gamma = 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^\Gamma$.

By definition,

$$e_i\left(C^*, x\right) = f_i\left(C^* \oplus_x (\{0\}, 0)\right) - f_i\left(C^* \oplus_0 (\{0\}, 0)\right).$$

Under SEP, it is straightforward to check that

$$f_i\left(C^\gamma(b) \oplus_0 (\{0\}, 0)\right) = f_i\left(C^{\gamma-1}(b) \oplus_{b_{\gamma-1}} (\{0\}, 0)\right)$$

for all $\gamma = \gamma_i + 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^{\Gamma}$. Now,

$$\begin{aligned}
& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) \\
= & \sum_{\gamma=\gamma_i}^{\Gamma} \left[f_i\left(C^{\gamma}\left(a'\right) \oplus_{a'_{\gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)\right] \\
= & f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) \\
= & \sum_{\gamma=\gamma_i}^{\Gamma} \left[f_i\left(C^{\gamma}\left(a\right) \oplus_{a_{\gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right)\right] \\
= & f_i\left(C^{\Gamma}\left(a\right) \oplus_{a_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) \\
= & f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right) \\
& + f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\Gamma}\left(a\right) \oplus_{a_{\Gamma}} \left(\{0\}, 0\right)\right)
\end{aligned}$$

Under SCM, $f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) \geq f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right)$.

We now prove that $f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right) = f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)$. For $\gamma_i = 1$, $C^1(a) = C^1(a') = C^1$ and the result holds trivially. Assume $\gamma_i > 2$. Then, $N^1 \cup ... \cup N^{\gamma_i - 1}$ and N^{γ_i} are two separable components in both $C^{\gamma_i}(a) \oplus_0 (\{0\}, 0)$ and $C^{\gamma_i}(a') \oplus_0 (\{0\}, 0)$. Moreover, the restriction of C^* to N^{γ_i} coincides in both *mcstp*. Under SEP, we obtain the result.

Hence,

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) \geq 0.$$

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