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Existence of financial equilibria in a multi-period stochastic economy

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Abstract. We consider the model of a stochastic financial exchange economy with finitely many periods. Time and uncertainty are represented by a finite event-tree $\mathbb{D}$ and consumers may have constraints on their portfolios. We provide a general existence result of financial equilibria, which allows to cover several important cases of financial structures in the literature with or without constraints on portfolios.

Key words: Incomplete markets, financial equilibria, constrained portfolios, multi-period model

1. Introduction

The main purpose of general equilibrium theory with incomplete markets is to study the interactions between the financial structure of the economy and the commodity structure, in a world in which time and uncertainty play a fundamental role. The first pioneering multiperiod model is due to Debreu ([10]), who introduced the idea of an event-tree of finite length, in order to represent time and uncertainty in a stochastic economy. Later, Magill and Schafer ([24]) extended the analysis of multi-period models, describing economies in which financial equilibria coincide with contingent market equilibria. The multi-period model was also explored, among others, by Duffie and Schafer

([12]), who proved a result of generic existence of equilibria, a detailed presentation of which is provided in Magill and Quinzii ([23]).

The multi-period model has been also extensively studied in the simple two-date model (one period $T = 1$): see, among others, [3, 26, 6], for the case of a finite set of states and [27, 28, 1, 30] for the case of a continuum of states. The two-date model, however, is not sufficient to capture the time evolution of realistic models. In this sense, the multiperiod model is much more flexible, and is also a necessary intermediate step before studying the infinite horizon setting (see [21, 22]). Moreover, multi-period models may provide a framework for phenomena which do not occur in a simple two-date model. For instance, in [4], Bonnisseau and Lachiri describe a three-date economy with production in which, essentially, the second welfare theorem does not hold, while it always holds in the two-date case. As a further example, we may recall that the suitable setting to study the effect of incompleteness of markets on price volatility is a three-date model, in the way addressed in [7].

In the model we consider, time and uncertainty are represented by an event-tree with T periods and finitely many nodes (date-events) at each date. At each node, there is a spot market where a finite set of commodities is available. Moreover, transfers of value among nodes and dates are made possible via a financial structure, namely finitely many financial assets available at each node of the event-tree. Our equilibrium notion encompasses the case in which retrading of financial assets is allowed at every node (see [23]) and we allow the case of restricted participation, namely the case in which agents' portfolio sets may be constrained.

This paper focuses on the existence of financial equilibria in a stochastic economy with general financial assets and possible constraints on portfolios. The existence problem with incomplete markets was studied, in the case of two-date models, by Cass ([5]) and Werner ([34, 35]), for nominal financial structures, Duffie ([11]) for purely financial securities under general conditions, Geanakoplos and Polemarchakis ([18]) in the case of numéraire assets. The existence of a financial equilibrium was proved by Bich and Cornet ([3]) when agents may have nontransitive preferences in the case of a two-date economy. In the case of T-period economies, we also mention the work by Duffie and Schafer ([13]) and by Florenzano and Gourdel ([15]); more recently, Da Rocha and Triki have studied a general intertemporal model in the case of purely financial securities ([25]). Other existence results in the infinite horizon models can be found in [20, 29, 16].

2. The T-period financial exchange economy

2.1 Time and uncertainty in a multi-period model

We[1] consider a multi-period exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, \ldots, T\}$, and a finite set of agents I. The stochastic structure of the model is described by a finite event-tree $\mathbb{D}$ of length T and we shall essentially use the same notations as [23] (we refer to [23] for an equivalent presentation with information partitions). The set $\mathbb{D}_t$ denotes the nodes (also called date-events) that could occur at date t and the family $(\mathbb{D}_t)_{t \in \mathcal{T}}$ defines a partition of the set $\mathbb{D}$; we denote by $t(\xi)$ the unique $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.

At each date $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date $t = 0$, which is denoted ξ_0, (or simply 0) so $\mathbb{D}_0 = \{\xi_0\}$. Finally, the event-tree $\mathbb{D}$ is endowed with a predecessor mapping $pr\colon\ \mathbb{D} \setminus \{\xi_0\} \longrightarrow \mathbb{D}$ which satisfies $pr(\mathbb{D}_t) = \mathbb{D}_{t-1}$, for every $t \neq 0$. The element $pr(\xi)$ is called the immediate predecessor of ξ and is also denoted ξ^-. For each $\xi \in \mathbb{D}$, we let $\xi^+ = \{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\}$ be the set of immediate successors of ξ; we notice that the set ξ^+ is nonempty if and only if $\xi \in \mathbb{D} \setminus \mathbb{D}_T$.

Moreover, for $\tau \in \mathcal{T} \setminus \{0\}$ and $\xi \in \mathbb{D} \setminus \bigcup_{t=0}^{\tau-1} \mathbb{D}_t$ we define, by induction, $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$ and we let the set of (not necessarily immediate) successors and the set of predecessors of ξ be respectively defined by

$$\mathbb{D}^+(\xi) = \{\xi' \in \mathbb{D} :\ \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau(\xi')\},$$
$$\mathbb{D}^-(\xi) = \{\xi' \in \mathbb{D} :\ \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\}.$$

If $\xi' \in \mathbb{D}^+(\xi)$ [resp. $\xi' \in \mathbb{D}^+(\xi) \cup \{\xi\}$], we shall also use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. We notice that $\mathbb{D}^+(\xi)$ is nonempty if and only if $\xi \notin \mathbb{D}_T$ and $\mathbb{D}^-(\xi)$ is nonempty if and only if $\xi \neq \xi_0$. Moreover, one has $\xi' \in \mathbb{D}^+(\xi)$ if and only if $\xi \in \mathbb{D}^-(\xi')$ (and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$).

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[1] In this paper, we shall use the following notations. A $(\mathbb{D} \times J)$-matrix A is an element of $\mathbb{R}^{\mathbb{D} \times J}$, with entries $(a(\xi, j))_{\xi \in \mathbb{D}, j \in J}$; we denote by $A(\xi) \in \mathbb{R}^J$ the ξ-th row of A and by $A(j) \in \mathbb{R}^{\mathbb{D}}$ the j-th column of A. We recall that the transpose of A is the unique $(J \times \mathbb{D})$-matrix ${}^tA$ satisfying $(Ax) \bullet_{\mathbb{D}} y = x \bullet_J ({}^tAy)$, for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\bullet_{\mathbb{D}}$ [resp. $\bullet_J$] denotes the usual scalar product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^J$]. We shall denote by $rank\ A$ the rank of the matrix A. For every subsets $\widetilde{\mathbb{D}} \subset \mathbb{D}$ and $\widetilde{J} \subset J$, the $(\widetilde{\mathbb{D}} \times \widetilde{J})$-sub-matrix of A is the $(\widetilde{\mathbb{D}} \times \widetilde{J})$-matrix $\widetilde{A}$ with entries $\widetilde{a}(\xi, j) = a(\xi, j)$ for every $(\xi, j) \in \widetilde{\mathbb{D}} \times \widetilde{J}$. Let x, y be in $\mathbb{R}^n$; we shall use the notation $x \geq y$ (resp. $x \gg y$) if $x_h \geq y_h$ (resp. $x_h \gg y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n :\ x \gg 0\}$. We shall also use the notation $x > y$ if $x \geq y$ and $x \neq y$. We shall denote by $\|\cdot\|$ the Euclidean norm in the different Euclidean spaces used in this paper and the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$ is denoted $B_L(x, r) := \{y \in \mathbb{R}^L :\ \|y - x\| \leq r\}$.

2.2 The stochastic exchange economy

At each node $\xi \in \mathbb{D}$, there is a spot market where a finite set H of divisible physical commodities is available. We assume that each commodity does not last for more than one period. In this model, a commodity is a couple (h, ξ) of a physical commodity $h \in H$ and a node $\xi \in \mathbb{D}$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L = H \times \mathbb{D}$. An element x in $\mathbb{R}^L$ is called a *consumption*, that is $x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^L$, where $x(\xi) = (x(h,\xi))_{h \in H} \in \mathbb{R}^H$, for every $\xi \in \mathbb{D}$.

We denote by $p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^L$ the vector of spot prices and $p(\xi) = (p(h,\xi))_{h \in H} \in \mathbb{R}^H$ is called the spot price at node ξ. The spot price $p(h,\xi)$ is the price paid, at date $t(\xi)$, for the delivery of one unit of commodity h at node ξ. Thus the value of the consumption $x(\xi)$ at node $\xi \in \mathbb{D}$ (evaluated in unit of account of node ξ) is

$$p(\xi) \bullet_H x(\xi) = \sum_{h \in H} p(h,\xi) x(h,\xi).$$

There is a finite set I of consumers and each consumer $i \in I$ is endowed with a *consumption set* $X^i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An *allocation* is an element $x \in \prod_{i \in I} X^i$, and we denote by x^i the consumption of agent i, that is the projection of x onto X^i.

The tastes of each consumer $i \in I$ are represented by a *strict preference correspondence* $P^i\colon \prod_{j \in I} X^j \longrightarrow X^i$, where $P^i(x)$ defines the set of consumptions that are strictly preferred by i to x^i, that is, given the consumptions x^j for the other consumers $j \neq i$. Thus P^i represents the tastes of consumer i but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers' preferences are represented by utility functions $u^i\colon X^i \longrightarrow \mathbb{R}$, for every $i \in I$, the strict preference correspondence is defined by $P^i(x) = \{\bar{x}^i \in X^i \mid u^i(\bar{x}^i) > u^i(x^i)\}$.

Finally, at each node $\xi \in \mathbb{D}$, every consumer $i \in I$ has a *node-endowment* $e^i(\xi) \in \mathbb{R}^H$ (contingent to the fact that ξ prevails) and we denote by $e^i = (e^i(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^L$ her *endowment vector* across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = \left[\mathbb{D}; H; I; (X^i, P^i, e^i)_{i \in I}\right].$$

2.3 The financial structure

We consider finitely many financial assets and we denote by J the set of assets. An asset $j \in J$ is a contract, which is issued at a given and unique node in $\mathbb{D}$, denoted by $\xi(j)$ and called the *emission node* of j. Each asset j is bought (or sold) at its emission node $\xi(j)$ and only yields payoffs at the successor nodes ξ' of $\xi(j)$, that is, for $\xi' > \xi(j)$. To allow for real assets, we let the payoff depend upon the spot price vector $p \in \mathbb{R}^L$ and we denote by $v(p, \xi, j)$ the payoff of asset j at node ξ. For the sake of convenient notations, we shall in fact consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and assume that it is zero if ξ is not a successor of the emission node $\xi(j)$. Formally, we assume that $v(p, \xi, j) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is $v(p, \xi_0, j) = 0$ for every $j \in J$; furthermore, every asset j which is emitted at the terminal date has a zero payoff, that is, if $\xi(j) \in \mathbb{D}_T$, $v(p, \xi, j) = 0$ for every $\xi \in \mathbb{D}$.

For every consumer $i \in I$, if $z^i_j > 0$ [resp. $z^i_j < 0$], then $|z^i_j|$ will denote the quantity of asset $j \in J$ bought [resp. sold] by agent i at the emission node $\xi(j)$. The vector $z^i = (z^i_j)_{j \in J} \in \mathbb{R}^J$ is called the *portfolio* of agent i.

We assume that each consumer $i \in I$ is endowed with a *portfolio set* $Z^i \subset \mathbb{R}^J$, which represents the set of portfolios that are admissible for agent i. This general framework allows us to treat, for example, the following important cases:

- $Z^i = \mathbb{R}^J$ (unconstrained portfolios);
- $Z^i \subset \underline{z}^i + \mathbb{R}^J_+$, for some $\underline{z}^i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales);
- $Z^i = B_J(0, 1)$ (bounded portfolios).

The price of asset j is denoted by q_j and we recall that it is paid at its emission node $\xi(j)$. We let $q = (q_j)_{j \in J} \in \mathbb{R}^J$ be the asset price (vector).

Definition 2.1. *A financial asset structure $\mathcal{F} = (J, (Z^i)_{i \in I}, (\xi(j))_{j \in J}, V)$ consists of*

– a set of assets J,

– a collection of portfolio sets $Z^i \subset \mathbb{R}^J$ for every agent $i \in I$,

– a node of issue $\xi(j) \in \mathbb{D}$ for each asset $j \in J$,

– a payoff mapping $V\colon \mathbb{R}^L \longrightarrow (\mathbb{R}^{\mathbb{D}})^J$ which associates, to every spot price $p \in \mathbb{R}^L$ the $(\mathbb{D} \times J)$-payoff matrix $V(p) = (v(p, \xi, j))_{\xi \in \mathbb{D}, j \in J}$, and satisfies the condition $v(p, \xi, j) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

The full matrix of payoffs *$W_{\mathcal{F}}(p, q)$ is the $(\mathbb{D} \times J)$-matrix with entries*

$$w_{\mathcal{F}}(p, q)(\xi, j) := v(p, \xi, j) - \delta_{\xi, \xi(j)} q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, for a given portfolio $z \in \mathbb{R}^J$ (and given prices (p, q)) the full flow of returns is $W_{\mathcal{F}}(p, q)z$ and the (full) financial return at node ξ is

$$\begin{aligned}[W_{\mathcal{F}}(p, q)z](\xi) := W_{\mathcal{F}}(p, q, \xi) \bullet_J z &= \sum_{j \in J} v(p, \xi, j) z_j - \sum_{j \in J} \delta_{\xi, \xi(j)} q_j z_j \\ &= \sum_{\{j \in J \mid \xi(j) < \xi\}} v(p, \xi, j) z_j - \sum_{\{j \in J \mid \xi(j) = \xi\}} q_j z_j,\end{aligned}$$

and we shall extensively use the fact that, for $\lambda \in \mathbb{R}^{\mathbb{D}}$, and $j \in J$, one has:

$$(2.1) \qquad [{}^t W_{\mathcal{F}}(p,q)\lambda](j) = \sum_{\xi \in \mathbb{D}} \lambda(\xi) v(p,\xi,j) - \sum_{\xi \in \mathbb{D}} \lambda(\xi)\delta_{\xi,\xi(j)}$$
$$= \sum_{\xi > \xi(j)} \lambda(\xi) v(p,\xi,j) - \lambda(\xi(j)) q_j.$$

In the following, when the financial structure $\mathcal{F}$ remains fixed, while only prices vary, we shall simply denote by $W(p,q)$ the full matrix of returns. In the case of unconstrained portfolios, namely $Z^i = \mathbb{R}^J$, for every $i \in I$, the financial asset structure will be simply denoted by $\mathcal{F} = (J, (\xi(j))_{j\in J}, V)$.

2.4 Financial equilibria

2.4.1 Financial equilibria without retrading

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$(\mathcal{E},\mathcal{F}) := \left[\mathbb{D}, H, I, (X^i, P^i, e^i)_{i\in I}; J, (Z^i)_{i\in I}, (\xi(j))_{j\in J}, V\right].$$

Given the price $(p,q) \in \mathbb{R}^L \times \mathbb{R}^J$, the *budget set* of consumer $i \in I$ is[2]

$$B^i_{\mathcal{F}}(p,q) = \big\{(x^i,z^i) \in X^i \times Z^i :$$
$$\forall \xi \in \mathbb{D},\ p(\xi) \bullet_H [x^i(\xi) - e^i(\xi)] \le [W_{\mathcal{F}}(p,q)z^i](\xi)\big\}$$
$$= \big\{(x^i,z^i) \in X^i \times Z^i :\ p \,\square\, (x^i - e^i) \le W_{\mathcal{F}}(p,q)z^i\big\}.$$

We now introduce the equilibrium notion.

Definition 2.2. *An* equilibrium *of the financial exchange economy* $(\mathcal{E},\mathcal{F})$ *is a list of strategies and prices* $(\bar{x},\bar{z},\bar{p},\bar{q}) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \setminus \{0\} \times \mathbb{R}^J$ *such that*
(a) for every $i \in I$, $(\bar{x}^i,\bar{z}^i)$ *maximizes the preferences* P^i *in the budget set* $B^i_{\mathcal{F}}(\bar{p},\bar{q})$, *in the sense that*

$$(\bar{x}^i,\bar{z}^i) \in B^i_{\mathcal{F}}(\bar{p},\bar{q}) \ \text{ and } \ [P^i(\bar{x}) \times Z^i] \cap B^i_{\mathcal{F}}(\bar{p},\bar{q}) = \emptyset;$$

(b) $\sum_{i\in I} \bar{x}^i = \sum_{i\in I} e^i$ *and* $\sum_{i\in I} \bar{z}^i = 0$.

In the Appendix we will show that the above definition is more general than the usual concept widely used in the literature (see for example Magill-Quinzii [23]). In particular, if we additionally assume that every asset of the financial structure $\mathcal{F}$ can be retraded at each node, the previous equilibrium notion coincides with the standard concept.

[2] For $x = (x(\xi))_{\xi\in\mathbb{D}}$, $p = (p(\xi))_{\xi\in\mathbb{D}}$ in $\mathbb{R}^L = \mathbb{R}^{H\times\mathbb{D}}$ (with $x(\xi)$, $p(\xi)$ in $\mathbb{R}^H$) we let $p \,\square\, x = (p(\xi) \bullet_H x(\xi))_{\xi\in\mathbb{D}} \in \mathbb{R}^{\mathbb{D}}$.

2.4.2 No-arbitrage and financial equilibria

When portfolios may be constrained, the concept of no-arbitrage has to be suitably modified. In particular, we shall make a distinction between the definitions of arbitrage-free portfolio and arbitrage-free financial structure.

Definition 2.3. *Given the financial structure* $\mathcal{F} = (J, (Z^i)_{i\in I}, (\xi(j))_{j\in J}, V)$, *the portfolio* $\bar{z}^i \in Z^i$ *is said to* have no arbitrage opportunities *or to be* arbitrage-free *for agent* $i \in I$ *at the price* $(p,q) \in \mathbb{R}^L \times \mathbb{R}^J$ *if there is no portfolio* $z^i \in Z^i$ *such that* $W_{\mathcal{F}}(p,q)z^i > W_{\mathcal{F}}(p,q)\bar{z}^i$, *that is,* $[W_{\mathcal{F}}(p,q)z^i](\xi) \geq [W_{\mathcal{F}}(p,q)\bar{z}^i](\xi)$, *for every* $\xi \in \mathbb{D}$, *with at least one strict inequality, or, equivalently, if*

$$W_{\mathcal{F}}(p,q)\,(Z^i - \bar{z}^i) \cap \mathbb{R}^{\mathbb{D}}_{+} = \{0\}.$$

The financial structure $\mathcal{F}$ *is said to be* arbitrage-free *at* (p,q) *if there exists no portfolios* $z^i \in Z^i$ $(i \in I)$ *such that* $W_{\mathcal{F}}(p,q)\big(\sum_{i\in I} z^i\big) > 0$, *or, equivalently, if:*

$$W_{\mathcal{F}}(p,q)\left(\sum_{i\in I} Z^i\right) \cap \mathbb{R}^{\mathbb{D}}_{+} = \{0\}.$$

Let the financial structure $\mathcal{F}$ be arbitrage-free at (p,q), and let $\bar{z}^i \in Z^i$ $(i \in I)$ such that $\sum_{i\in I} \bar{z}^i = 0$, then it is easy to see that, for every $i \in I$, $\bar{z}^i$ is arbitrage-free at (p,q). The converse is true, for example, when some agent's portfolio set is unconstrained, that is, when $Z^i = \mathbb{R}^J$ for some $i \in I$.

We recall that equilibrium portfolios are arbitrage-free under the following Non-Satiation Assumption:

Assumption NS (*i*) For every $\bar{x} \in \prod_{i\in I} X^i$ such that $\sum_{i\in I} \bar{x}^i = \sum_{i\in I} e^i$, (Non-Satiation at Every Node) for every $\xi \in \mathbb{D}$, there exists $x \in \prod_{i\in I} X^i$ such that, for each $\xi' \neq \xi$, $x^i(\xi') = \bar{x}^i(\xi')$ and $x^i \in P^i(\bar{x})$;

(*ii*) if $x^i \in P^i(\bar{x})$, then $[x^i, \bar{x}^i[\, \subset P^i(\bar{x})$.

Proposition 2.1. *Under (NS), if* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *is an equilibrium of the economy* $(\mathcal{E}, \mathcal{F})$, *then* $\bar{z}^i$ *is arbitrage-free at* $(\bar{p}, \bar{q})$ *for every* $i \in I$.

Proof. By contradiction. If, for some $i \in I$, the portfolio $\bar{z}^i$ is not arbitrage-free at $(\bar{p}, \bar{q})$, then there exists $z^i \in Z^i$ such that $W_{\mathcal{F}}(\bar{p},\bar{q})z^i > W_{\mathcal{F}}(\bar{p},\bar{q})\bar{z}^i$, namely $[W_{\mathcal{F}}(\bar{p},\bar{q})z^i](\xi) \geq [W_{\mathcal{F}}(\bar{p},\bar{q})\bar{z}^i](\xi)$, for every $\xi \in \mathbb{D}$, with at least one strict inequality, say for $\bar{\xi} \in \mathbb{D}$.

Since $\sum_{i\in I}(\bar{x}^i - e^i) = 0$, from Assumption (NS.i), there exists $x \in \prod_{i\in I} X^i$ such that, for each $\xi \neq \bar{\xi}$, $x^i(\xi) = \bar{x}^i(\xi)$ and $x^i \in P^i(\bar{x})$. Let us consider $\lambda \in\,]0,1[$ and define $x^i_\lambda := \lambda x^i + (1-\lambda)\bar{x}^i$; then, by Assumption (NS.ii), $x^i_\lambda \in\,]x^i, \bar{x}^i[\, \subset P^i(\bar{x})$.

In the following, we prove that, for $\lambda > 0$ small enough, $(x^i_\lambda, z^i) \in B^i_{\mathcal{F}}(\bar{p},\bar{q})$, which will contradict the fact that $[P^i(\bar{x}) \times Z^i] \cap B^i_{\mathcal{F}}(\bar{p},\bar{q}) = \emptyset$

(since $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium). Indeed, since $(\bar{x}^i, \bar{z}^i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, for every $\xi \neq \bar{\xi}$ we have:

$$\bar{p}(\xi) \bullet_H [x^i_\lambda(\xi) - e^i(\xi)] = \bar{p}(\xi) \bullet_H [\bar{x}^i(\xi) - e^i(\xi)] \\ \leq [W_{\mathcal{F}}(\bar{p}, \bar{q})\bar{z}^i](\xi) \leq [W_{\mathcal{F}}(\bar{p}, \bar{q})z^i](\xi).$$

Now, for $\xi = \bar{\xi}$, we have

$$\bar{p}(\bar{\xi}) \bullet_H [\bar{x}^i(\bar{\xi}) - e^i(\bar{\xi})] \leq [W_{\mathcal{F}}(\bar{p}, \bar{q})\bar{z}^i](\bar{\xi}) < [W_{\mathcal{F}}(\bar{p}, \bar{q})z^i](\bar{\xi}).$$

But, when $\lambda \to 0$, $x^i_\lambda \to \bar{x}^i$, hence for $\lambda > 0$ small enough we have

$$\bar{p}(\bar{\xi}) \bullet_H [x^i_\lambda(\bar{\xi}) - e^i(\bar{\xi})] < [W_{\mathcal{F}}(\bar{p}, \bar{q})z^i](\bar{\xi}).$$

Consequently, $(x^i_\lambda, z^i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$. □

2.4.3 A characterization of no-arbitrage with constrained portfolio sets

When the portfolios sets may be constrained, the following theorem extends the standard characterization result of no-arbitrage in terms of state prices.

Theorem 2.1. *Let $\mathcal{F} = (J, (Z^i)_{i \in I}, (\xi(j))_{j \in J}, V)$, let $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, for $i \in I$, let $z^i \in Z^i$, assume that Z^i is convex and consider the following statements:*

(i) there exists $\lambda^i = (\lambda^i(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW_{\mathcal{F}}(p, q)\lambda^i \in N_{Z^i}(z^i)$,[3] *or, equivalently, there exists $\eta \in N_{Z^i}(z^i)$ such that:*

$$\lambda^i(\xi(j))q_j = \sum_{\xi > \xi(j)} \lambda^i(\xi) v(p, \xi, j) - \eta_j \ \textit{for every} \ j \in J;$$

(ii) the portfolio z^i is arbitrage-free for agent $i \in I$ at (p, q).
The implication [(i) $\Rightarrow$ (ii)] always holds and the converse is true under the additional assumption that Z^i is a polyhedral set[4].

The above Theorem 2.1 is a consequence of Theorem 5.1, stated and proved in the Appendix, the main part (i.e., the existence of positive node prices $\lambda^i(\xi)$) being due to Koopmans [19].

[3] We recall that $N_{Z^i}(z^i)$ is the normal cone to Z^i at z^i, which is defined as $N_{Z^i}(z^i) := \{\eta \in \mathbb{R}^J : \eta \bullet_J z^i \geq \eta \bullet_J (z')^i, \forall (z')^i \in Z^i\}$.

[4] A subset $C \subset \mathbb{R}^n$ is said to be *polyhedral* if it is the intersection of finitely many closed half-spaces, namely $C = \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is a real $(m \times n)$-matrix, and $b \in \mathbb{R}^m$. Note that polyhedral sets are always closed and convex and that the empty set and the whole space $\mathbb{R}^n$ are both polyhedral.

3. Existence of equilibria

3.1 The main existence result

Our main existence result allows agents to have constrained portfolios, that is, we do not assume that $Z^i = \mathbb{R}^J$. We shall allow the financial structure to be general enough to cover important cases such as bounded assets (as in Radner [32]) and nominal assets; our approach however does not cover the general case of real assets which needs a different and specific treatment. Let us consider the financial economy

$$(\mathcal{E}, \mathcal{F}) = \left[\mathbb{D}, H, I, (X^i, P^i, e^i)_{i\in I}; J, (Z^i)_{i\in I}, (\xi(j))_{j\in J}, V\right].$$

We introduce the following assumptions:

Assumption (C) (Consumption Side) *For all $i \in I$ and all $\bar{x} \in \prod_{i\in I} X^i$,*

(i) X^i is a closed and convex subset of $\mathbb{R}^L$;
(ii) the preference correspondence P^i, from $\prod_{i\in I} X^i$ to X^i, is lower semicontinuous[5] and $P^i(\bar{x})$ is convex;
(iii) for every $x^i \in P^i(\bar{x})$ for every $(x')^i \in X^i$, $(x')^i \neq x^i$, $[(x')^i, x^i[\cap P^i(\bar{x}) \neq \emptyset$;[6]
(iv) (Irreflexivity) $\bar{x}^i \notin P^i(\bar{x})$;
(v) (Non-Satiation of Preferences at Every Node) if $\sum_{i\in I} \bar{x}^i = \sum_{i\in I} e^i$, for every $\xi \in \mathbb{D}$ there exists $x \in \prod_{i\in I} X^i$ such that, for each $\xi' \neq \xi$, $x^i(\xi') = \bar{x}^i(\xi')$ and $x^i \in P^i(\bar{x})$;
(vi) (Strong Survival Assumption) $e^i \in \operatorname{int} X^i$.

Assumption (F) (Financial Side)

(i) The application $p \mapsto V(p)$ is continuous;
(ii) for every $i \in I$, Z^i is a closed, convex subset of $\mathbb{R}^J$ containing 0;
(iii) there exists $i_0 \in I$ such that $0 \in \operatorname{int} Z^{i_0}$.

We now state the last assumption for which we need to define the set of admissible consumptions and portfolios for a fixed $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, that is,

$$\begin{aligned} B(\lambda) := \Big\{ &(x, z) \in \textstyle\prod_{i\in I} X^i \times \prod_{i\in I} Z^i : \exists (p, q) \in B_L(0,1) \times \mathbb{R}^J, \\ &{}^tW_{\mathcal{F}}(p,q)\lambda \in B_J(0,1),\ (x^i, z^i) \in B^i_{\mathcal{F}}(p,q) \text{ for every } i \in I, \\ &\textstyle\sum_{i\in I} x^i = \sum_{i\in I} e^i,\ \sum_{i\in I} z^i = 0 \Big\}. \end{aligned}$$

[5] A correspondence $\varphi\colon X \longrightarrow Y$ is said to be lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \varphi(x_0)$ is not empty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \varphi(x)$ is nonempty. The correspondence φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X.

[6] This is satisfied, in particular, when $P^i(\bar{x})$ is open in X^i (for its relative topology).

Boundedness Assumption (B_λ) *The set $B(\lambda)$ is bounded.*

In the next section Assumption (B_λ) will be discussed and we will give different important cases in which it is satisfied.

Theorem 3.1. *(a) Let $(\mathcal{E}, \mathcal{F})$ be a financial economy satisfying Assumptions* (C), (F), *let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ satisfying (B_λ), and let $i_0 \in I$ be some agent such that $0 \in \operatorname{int} Z^{i_0}$. Then there exists an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $(\mathcal{E}, \mathcal{F})$ such that, for every $\xi \in \mathbb{D}$, $\bar{p}(\xi) \neq 0$ and*

$$^tW_{\mathcal{F}}(\bar{p}, \bar{q})\lambda \in N_{Z^{i_0}}(\bar{z}^{i_0}),$$

or, equivalently, there exists $\bar{\eta} \in N_{Z^{i_0}}(\bar{z}^{i_0})$ such that

$$\lambda(\xi(j))\bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) v(\bar{p}, \xi, j) - \bar{\eta}_j \text{ for every } j \in J.$$

(b) If moreover $\bar{z}^{i_0} \in \operatorname{int} Z^{i_0}$, then $^tW_{\mathcal{F}}(\bar{p}, \bar{q})\lambda = 0$, or, equivalently,

$$\lambda(\xi(j))\bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) v(\bar{p}, \xi, j) \text{ for every } j \in J,$$

hence the financial structure $\mathcal{F}$ is arbitrage-free at $(\bar{p}, \bar{q})$.

The proof of Theorem 3.1 will be given in the following section. From Theorem 3.1 we deduce directly the standard existence result in the case of unconstrained portfolios.

Corollary 3.1. *[Unconstrained portfolio case] Let $(\mathcal{E}, \mathcal{F})$ be a financial economy and let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be such that Assumptions* (C), (F) *and (B_λ) hold and $Z^i = \mathbb{R}^J$ for some $i \in I$. Then $(\mathcal{E}, \mathcal{F})$ admits an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \prod_{i\in I} X^i \times \prod_{i \in I} Z^i \times \mathbb{R}^L \times \mathbb{R}^J$ such that, for every $\xi \in \mathbb{D}$, $\bar{p}(\xi) \neq 0$ and*

$$^tW(\bar{p}, \bar{q})\lambda = 0,$$

or, equivalently,

$$\lambda(\xi(j))\bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) v(\bar{p}, \xi, j) \text{ for every } j \in J.$$

3.2 Existence for various financial models

We first state a proposition giving sufficient conditions for Assumption (B_λ) to hold. We recall that an asset j is said to be *short-lived*, when the payoffs are paid only at the immediate successors of its emission node, that is, formally, for every spot price $p \in \mathbb{R}^L$, $v(p, \xi, j) = 0$ if $\xi \notin \xi(j)^+$. An asset is said to be *long-lived* if it is not short-lived. A financial structure is said to be *short-lived* if all its assets are short-lived; it is said to be *long-lived* if it is not short-lived.

Proposition 3.1. *Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be fixed and assume that, for every $i \in I$, X^i is bounded from below. Then Assumption (B_λ) is satisfied if one of the following conditions holds:*

(i) [Bounded Below Portfolios] for every $i \in I$, the portfolio set Z^i is bounded from below, namely there exists $\underline{z}^i \in -\mathbb{R}^J_+$ such that $Z^i \subset \underline{z}^i + \mathbb{R}^J_+$;
(ii) [Rank Condition for Long-Lived Assets] for every $(p, q, \eta) \in B_L(0,1) \times \mathbb{R}^J \times B_J(0,1)$ such that ${}^tW(p,q)\lambda = \eta$, then rank $W(p,q) = \#J$.
(iii) [Rank Condition for Short-Lived Assets] $\mathcal{F}$ consists only of short-lived assets and rank $V(p) = \#J$ for every $p \in \mathbb{R}^L$.

The proof of Proposition 3.1 is given in the Appendix.

We now deduce from Proposition 3.1 and the main existence Theorem 3.1, the following existence result of equilibria in the case of bounded portfolios due to Radner [32].

Corollary 3.2. *[Bounded from below portfolio sets] Let $(\mathcal{E}, \mathcal{F})$ and $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ satisfy Assumptions* (C), (F) *and assume that, for every $i \in I$, X^i is bounded from below and $Z^i \subset \underline{z}^i + \mathbb{R}^J_{++}$, where $\underline{z}^i \in -\mathbb{R}^J_+$. Then there exists an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \prod_{i\in I} X^i \times \prod_{i \in I} Z^i \times \mathbb{R}^L \times \mathbb{R}^J$ of $(\mathcal{E}, \mathcal{F})$, such that, for every $\xi \in \mathbb{D}$, $\bar{p}(\xi) \neq 0$ and*

$${}^tW(\bar{p}, \bar{q})\lambda \leq 0 \text{ and the equality holds for each component } j \text{ such that } \bar{z}^i_j > \underline{z}^i_j,$$

or, equivalently,

$$\text{for every } j \in J,\ \lambda(\xi(j))\bar{q}_j \geq \sum_{\xi > \xi(j)} \lambda(\xi) v(\bar{p}, \xi, j), \text{ with equality if } \bar{z}^i_j > \underline{z}^i_j.$$

We end this section with the case of short-lived assets, which is a natural generalization of the classical two-date model ($T = 1$) that has been extensively studied in the literature due to its simple tractability (see the Appendix for several important properties of the two-date model that are still valid in the case of short-lived financial structures).

Corollary 3.3. *[Short-lived nominal assets] Let us assume that the economy $(\mathcal{E}, \mathcal{F})$ satisfies Assumption* (C), *X^i is bounded from below, for every $i \in I$, $\mathcal{F}$ consists of nominal short-lived assets and assume that one of the following conditions holds:*

(i) [unconstrained case] $Z^i = \mathbb{R}^J$ for every $i \in I$;

(ii) [constrained case] Z^i is a closed and convex subset of $\mathbb{R}^J$ containing 0; $0 \in \text{int}\, Z^{i_0}$ *for some $i_0 \in I$; rank $V = \sharp J$.*

For every $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, $(\mathcal{E}, \mathcal{F})$ admits an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \prod_{i\in I} X^i \times \prod_{i\in I} Z^i \times \mathbb{R}^L \times \mathbb{R}^J$ such that, for every $\xi \in \mathbb{D}$, $\bar{p}(\xi) \neq 0$ and $\bar{q}$ is the no-arbitrage price associated to λ, that is

$$^tW(\bar{q})\lambda \in N_{Z^{i_0}}(\bar{z}^{i_0}) \quad (resp.\ ^tW(\bar{q})\lambda = 0,\ under\ (i)),$$

or, equivalently, there exists $\bar{\eta} \in N_{Z^{i_0}}(\bar{z}^{i_0})$ *(resp.* $\bar{\eta} = 0$*, under (i)) such that*

$$\lambda(\xi(j))\bar{q}_j = \sum_{\xi \in \xi(j)^+} \lambda(\xi)v(\xi, j) - \bar{\eta}_j \text{ for every } j \in J.$$

Proof. Let $r := rank\, V$. We can define a new financial structure $\mathcal{F}'$ with r nominal assets by eliminating the redundant assets. Formally, we let $J' \subset J$ be the set of r assets such that the columns $(V(j))_{j \in J'}$ are independent and V' the associated return matrix. The new financial structure is

$$\mathcal{F}' := (J', (\xi(j))_{j \in J'}, V').$$

Then $rank\, W_{\mathcal{F}'}(q) = r$ since, by Proposition 5.2, $r = rank\, V' \leq rank\, W_{\mathcal{F}'}(q)$ $(\leq \min\{r, \mathbb{D}\})$. Consequently, by Proposition 3.1, the set $B(\lambda)$ is bounded.

From the existence theorem (Corollary 3.1), for every $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ there exists an equilibrium $(\bar{x}, z', \bar{p}, q')$ of $(\mathcal{E}, \mathcal{F}')$ (where $q' \in \mathbb{R}^{J'}$ and $z' \in (\mathbb{R}^{J'})^I$) such that $^tW_{\mathcal{F}'}(q')\lambda = 0$ or, equivalently,

$$\lambda(\xi(j))q'_j = \sum_{\xi' \in \xi(j)^+} \lambda(\xi')v(\xi', \xi(j)).$$

for every $j \in J'$. Now it is easy to see that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$, by defining $\bar{q} \in \mathbb{R}^J$ as $^tW(\bar{q})\lambda = 0$, that is

$$\lambda(\xi(j))\bar{q}_j = \sum_{\xi' \in \xi(j)^+} \lambda(\xi')v(\xi', \xi(j)),$$

for every $j \in J$, and $\bar{z}^i \in \mathbb{R}^J$ as $\bar{z}^i_j = z'^i_j$, if $j \in J'$, and $\bar{z}^i_j = 0$, if $j \in J \setminus J'$. □

4. Proof of the main result

4.1 Proof under additional assumptions

In this section, we shall prove Theorem 3.1 under the additional assumption
Assumption (K) *For every* $i \in I$,

(i) X^i *and* Z^i *are compact;*
(ii) [Local Non-Satiation] for every $\bar{x} \in \prod_{i \in I} X^i$*, for every* $x^i \in P^i(\bar{x})$ *then* $[x^i, \bar{x}^i[\subset P^i(\bar{x})$.

4.1.1 Preliminary definitions

In the following we fix some agent i_0, say $i_0 = 1$, for whom the assumption $0 \in \operatorname{int} Z^{i_0}$ is satisfied and we fix $\lambda = (\lambda(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$. We recall that for $(p, \eta) \in \mathbb{R}^L \times \mathbb{R}^J$, the vector $q = q(p, \eta) \in \mathbb{R}^J$ is uniquely defined by the equation

$$ {}^t W_{\mathcal{F}}(p, q)\lambda - \eta = 0, $$

which, from Theorem 2.1, is equivalent to saying that

$$ q_j(p, \eta) = \frac{1}{\lambda(\xi(j))} \left(\sum_{\xi > \xi(j)} \lambda(\xi) v(p, \xi, j) - \eta_j \right) \quad \text{for every } j \in J, $$

and, from Assumption (F), the mapping $(p, \eta) \mapsto q(p, \eta)$ is continuous. For (p, η) in the set $B := \{(p, \eta) \in \mathbb{R}^L \times \mathbb{R}^J : \|\lambda \square p\| \leq 1, \|\eta\| \leq 1\}$, we define

$$ \rho(p, \eta) = \max\{0, 1 - \|\lambda \square p\| - \|\eta\|\}. $$

Following the so-called Cass' trick, hereafter, we shall distinguish Consumer 1 from the other agents, and we shall extend the budget sets as in Bergstrom ([2]). In the following, we let $\mathbb{1} = (1, \dots, 1)$ denote the element in $\mathbb{R}^{\mathbb{D}}$, whose coordinates are all equal to one. For $(p, \eta) \in B$, we define the following augmented budget sets: first, for $i = 1$,

$$ \beta^1(p, \eta) = \left\{ x^1 \in X^1 : (\lambda \square p) \bullet_L (x^1 - e^1) \leq \sup_{z \in Z^1} \eta \bullet_J z + \rho(p, \eta) \sum_{\xi \in \mathbb{D}} \lambda(\xi) \right\}, $$

$$ \alpha^1(p, \eta) = \left\{ x^1 \in X^1 : (\lambda \square p) \bullet_L (x^1 - e^1) < \sup_{z \in Z^1} \eta \bullet_J z + \rho(p, \eta) \sum_{\xi \in \mathbb{D}} \lambda(\xi) \right\}, $$

and for $i \neq 1$

$$ \beta^i(p, \eta) = \left\{ (x^i, z^i) \in X^i \times Z^i : p \square (x^i - e^i) \leq W_{\mathcal{F}}(p, q(p, \eta)) z^i + \rho(p, \eta) \mathbb{1} \right\}, $$
$$ \alpha^i(p, \eta) = \left\{ (x^i, z^i) \in X^i \times Z^i : p \square (x^i - e^i) \ll W_{\mathcal{F}}(p, q(p, \eta)) z^i + \rho(p, \eta) \mathbb{1} \right\}. $$

We now define the following enlarged set of agents denoted I_0, by considering all the agents in $i \in I \backslash \{1\}$, by counting twice the agent 1, denoted by $i = (1, 1)$ and $i = (1, 2)$ and by considering an additional agent denoted $i = 0$. The additional and fictitious agent $i = 0$ is traditional and will fix the equilibrium prices $(\bar{p}, \bar{q})$ and the agent $i = 1$ has been disaggregated so that $i = (1, 1)$ will fix the equilibrium consumption $\bar{x}^1$ and $i = (1, 2)$ will fix the equilibrium portfolio $\bar{z}^1$ (which thus can be chosen by two independent maximization problems). For $(x, z, (p, \eta)) \in \prod_{i \in I} X^i \times \prod_{i \in I} Z^i \times B$, we define the correspondences Φ^i for $i \in I_0$ as follows:

$$\Phi^0(x,z,(p,\eta))$$
$$= \left\{ (p',\eta') \in B \;\middle|\; \sum_{\xi\in\mathbb{D}} \left[\lambda(\xi)(p'(\xi)-p(\xi)) \bullet_H \sum_{i\in I}(x^i(\xi)-e^i(\xi)) \right] - (\eta'-\eta)\bullet_J \sum_{i\in I} z^i > 0 \right\},$$

$$\Phi^{1,1}(x,z,(p,\eta)) = \begin{cases} \beta^1(p,\eta) & \text{if } x^1 \notin \beta^1(p,\eta), \\ \alpha^1(p,\eta)\cap P^1(x) & \text{if } x^1 \in \beta^1(p,\eta), \end{cases}$$

$$\Phi^{1,2}(x,z,p,\eta) = \left\{ z'^1 \in Z^1 \mid \eta \bullet_J z'^1 > \eta \bullet_J z^1 \right\},$$

and for every $i \in I$, $i \neq 1$

$$\Phi^i(x,z,(p,\eta)) = \begin{cases} \{(e^i,0)\} & \text{if } (x^i,z^i)\notin \beta^i(p,\eta) \text{ and } \alpha^i(p,\eta)=\emptyset, \\ \beta^i(p,\eta) & \text{if } (x^i,z^i)\notin \beta^i(p,\eta) \text{ and } \alpha^i(p,\eta)\neq\emptyset, \\ \alpha^i(p,\eta)\cap(P^i(x)\times Z^i) & \text{if } (x^i,z^i)\in \beta^i(p,\eta). \end{cases}$$

4.1.2 The fixed-point argument

The existence proof relies on the following fixed-point-type theorem due to Gale and Mas Colell ([17]).

Theorem 4.1. *Let I_0 be a finite set, let C^i ($i \in I_0$) be a nonempty, compact, convex subset of some Euclidean space, let $C = \prod_{i\in I} C^i$ and let Φ^i ($i \in I_0$) be a correspondence from C to C^i, which is lower semicontinuous and convex-valued. Then, there exists $\bar{c} \in C$ such that, for every $i \in I_0$ [either $\bar{c}^i \in \Phi^i(\bar{c})$ or $\Phi^i(\bar{c}) = \emptyset$].*

We now show that, for $i \in I_0$, the sets $C^0 = B$, $C^{1.1} = X^1$, $C^{1.2} = Z^1$, $C^i = X^i \times Z^i$ and the above defined correspondences Φ^i ($i \in I_0$) satisfy the assumptions of Theorem 4.1.

Claim 4.1. *For every $\bar{c} := (\bar{x},\bar{z},(\bar{p},\bar{\eta})) \in \prod_{i\in I} X^i \times \prod_{i\in I} Z^i \times B$, for every $i \in I_0$, the correspondence Φ^i is lower semicontinuous at $\bar{c}$, the set $\Phi^i(\bar{c})$ is convex (possibly empty) and $(\bar{p},\bar{\eta}) \notin \Phi^0(\bar{c})$, $\bar{x}^1 \notin \Phi^{1.1}(\bar{c})$, $\bar{z}^1 \notin \Phi^{1.2}(\bar{c})$, $(\bar{x}^i,\bar{z}^i) \notin \Phi^i(\bar{c})$ for $i > 1$.*

Proof. Let $\bar{c} := (\bar{x},\bar{z},(\bar{p},\bar{\eta})) \in \prod_{i\in I} X^i \times \prod_{i\in I} Z^i \times B$ be given. We first notice that $\Phi^i(\bar{c})$ is convex for every $i \in I_0$, recalling that $P^i(\bar{x})$ is convex, by Assumption (C). Clearly, $(\bar{p},\bar{\eta}) \notin \Phi^0(\bar{c})$ and $\bar{z}^1 \notin \Phi^{1.2}(\bar{c})$ from the definition of these two sets; the two last properties $\bar{x}^1 \notin \Phi^{1.1}(\bar{c})$ and $(\bar{x}^i,\bar{z}^i) \notin \Phi^i(\bar{c})$ follow from the definitions of these sets and the fact that $\bar{x}^i \notin P^i(\bar{x})$ from Assumption (C).

We now show that Φ^i is lower semicontinuous at $\bar{c}$.

Step 1: $i \in I$, $i > 1$. Let U be an open subset of $X^i \times Z^i$ such that $\Phi^i(\bar{c}) \cap U \neq \emptyset$. We will distinguish three cases:
Case (i): $(\bar{x}^i, \bar{z}^i) \notin \beta^i(\bar{p}, \bar{\eta})$ and $\alpha^i(\bar{p}, \bar{\eta}) = \emptyset$. Then $\Phi^i(\bar{c}) = \{(e^i, 0)\} \subset U$. Since the set $\{(x^i, z^i, (p, \eta)) \mid (x^i, z^i) \notin \beta^i(p, \eta)\}$ is an open subset of $X^i \times Z^i \times B$ (by Assumptions (C) and (F)), it contains an open neighborhood O of $\bar{c}$. Now, let $c = (x, z, (p, \eta)) \in O$. If $\alpha^i(p, \eta) = \emptyset$ then $\Phi^i(c) = \{(e^i, 0)\} \subset U$ and so $\Phi^i(c) \cap U$ is nonempty. If $\alpha^i(p, \eta) \neq \emptyset$ then $\Phi^i(c) = \beta^i(p, \eta)$. But Assumptions (C) and (F) imply that $(e^i, 0) \in X^i \times Z^i$, hence $(e^i, 0) \in \beta^i(p, \eta)$ (noticing that $\rho(p, q) \geq 0$). So $\{(e^i, 0)\} \subset \Phi^i(c) \cap U$ which is also nonempty.
Case (ii): $\bar{c} = (\bar{x}^i, \bar{z}^i, (\bar{p}, \bar{\eta})) \in \Omega^i := \{c = (x^i, z^i, (p, \eta)) : (x^i, z^i) \notin \beta^i(p, \eta)$ and $\alpha^i(p, \eta) \neq \emptyset\}$. Then the set Ω^i is clearly open and on the set Ω^i one has $\Phi^i(c) = \beta^i(p, \eta)$. We recall that $\emptyset \neq \Phi^i(\bar{c}) \cap U = \beta^i(\bar{p}, \bar{\eta}) \cap U$. We notice that $\beta^i(\bar{p}, \bar{\eta}) = \operatorname{cl} \alpha^i(\bar{p}, \bar{\eta})$ since $\alpha^i(\bar{p}, \bar{\eta}) \neq \emptyset$. Consequently, $\alpha^i(\bar{p}, \bar{\eta}) \cap U \neq \emptyset$ and we choose a point $(x^i, z^i) \in \alpha^i(\bar{p}, \bar{\eta}) \cap U$, that is, $(x^i, z^i) \in [X^i \times Z^i] \cap U$ and

$$\bar{p} \,\square\, (x^i - e^i) \ll W_{\mathcal{F}}(\bar{p}, q(\bar{p}, \bar{\eta}))z^i + \rho(\bar{p}, \bar{\eta})\mathbb{1}.$$

Clearly the above inequality is also satisfied for the same point point (x^i, z^i) when (p, η) belongs to a neighborhood O of $(\bar{p}, \bar{\eta})$ small enough (using the continuity of $q(\cdot, \cdot)$ and $p(\cdot, \cdot)$). This shows that on O one has $\emptyset \neq \alpha^i(p, \eta) \cap U \subset \beta^i(p, \eta) \cap U = \Phi(c) \cap U$.
Case (iii): $(\bar{x}^i, \bar{z}^i) \in \beta^i(\bar{p}, \bar{\eta})$. By assumption we have

$$\emptyset \neq \Phi^i(\bar{c}) \cap U = \alpha^i(\bar{p}, \bar{q}) \cap [P^i(\bar{x}) \times Z^i] \cap U.$$

By an argument similar to what is done above, one shows that there exists an open neighborhood N of $(\bar{p}, \bar{q})$ and an open set M such that, for every $(p, \eta) \in N$, one has $\emptyset \neq M \subset \alpha^i(p, \eta) \cap U$. Since P^i is lower semicontinuous at $\bar{x}$ (by Assumption (C)), there exists an open neighborhood Ω of $\bar{x}$ such that, for every $x \in \Omega$, $\emptyset \neq [P^i(x) \times Z^i] \cap M$, hence

$$\emptyset \neq [P^i(x) \times Z^i] \cap \alpha^i(p, \eta) \cap U \subset \beta^i(p, \eta) \cap U, \quad \text{for every } x \in \Omega.$$

Consequently, from the definition of Φ^i, we get $\emptyset \neq \Phi^i(c) \cap U$, for every $c \in \Omega$.

The correspondence $\Psi^i := \alpha^i \cap (P^i \times Z^i)$ is lower semicontinuous on the whole set, being the intersection of an open graph correspondence and a lower semicontinuous correspondence. Then there exists an open neighborhood O of $\bar{c} := (\bar{x}, \bar{z}, (\bar{p}, \bar{\eta}))$ such that, for every $(x, z, (p, \eta)) \in O$, then $U \cap \Psi^i(x, z, (p, \eta)) \neq \emptyset$ hence $\emptyset \neq U \cap \Phi^i(x, z, (p, \eta))$ (since we always have $\Psi^i(x, z, (p, \eta)) \subset \Phi^i(x, z, (p, \eta)))$.

Step 2: $i = (1, 1)$. The proof is similar to the first step and more standard. We only check hereafter that the case $\alpha^1(p, \eta) = \emptyset$ never holds. Indeed, we will consider three cases. If $\eta \neq 0$ then $0 < \max\{\eta \bullet_J z^1 \mid z^1 \in Z^1\}$ since

$0 \in \operatorname{int} Z^1$ (by Assumption (F)). So $e^1 \in \alpha^1(p,\eta)$ since $e^1 \in X^1$ (by Assumption (C)). If $\eta = 0$ and $p = 0$, then $\rho(p,\eta) = 1$ and again $e^1 \in \alpha^1(p,\eta)$. Finally, if $\eta = 0$ and $p \neq 0$, then $e^1 - t(\lambda \,\square\, p) \in \alpha^1(p,\eta)$ for $t > 0$ small enough since $e^1 \in \operatorname{int} X^1$ (by Assumption (C)).

Step 3: $i = 0$ and $i = (1,2)$. Obvious. □

For $i = 0$, for every $(p,\eta) \in B$, in view of Claim 4.1, we can now apply the fixed-point Theorem 4.1. Hence there exists $\bar{c} := (\bar{x}, \bar{z}, (\bar{p}, \bar{\eta})) \in \prod_{i\in I} X^i \times \prod_{i \in I} Z^i \times B$ such that, for every $i \in I_0$, $\Phi^i(\bar{x}, \bar{z}, (\bar{p}, \bar{\eta})) = \emptyset$. Written coordinatewise, this is equivalent to saying that:

$$(4.1)\qquad (\lambda \,\square\, p) \bullet_L \sum_{i\in I} (\bar{x}^i - e^i) - \eta \bullet_J \sum_{i \in I} \bar{z}^i \\ \leq (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i\in I} (\bar{x}^i(\xi) - e^i(\xi)) - \bar{\eta} \bullet_J \sum_{i\in I} \bar{z}^i,$$

for $i = (1,1)$

$$(4.2)\qquad \bar{x}^1 \in \beta^1(\bar{p}, \bar{\eta}) \text{ and } \alpha^1(\bar{p},\bar{\eta}) \cap P^1(\bar{x}) = \emptyset,$$

for $i = (1,2)$

$$(4.3)\qquad \bar{\eta} \bullet_J \bar{z}^1 = \max\{\bar{\eta} \bullet_J z^1 \mid z^1 \in Z^1\};$$

for the remaining i

$$(4.4)\qquad (\bar{x}^i, \bar{z}^i) \in \beta^i(\bar{p},\bar{\eta}) \text{ and } \alpha^i(\bar{p},\bar{\eta}) \cap (P^i(\bar{x}) \times Z^i) = \emptyset.$$

From now on we shall denote simply by W the full matrix of returns $W_{\mathcal{F}}(\bar{p}, \bar{q})$ associated to the spot price $\bar{p}$ and to the asset price $\bar{q} = q(\bar{p}, \bar{\eta})$.

4.1.3 The vector $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium

We recall that, from Theorem 2.1, $\bar{q} = q(\bar{p},\bar{\eta})$ is the unique vector $\bar{q} \in \mathbb{R}^J$ satisfying

$${}^tW\lambda = \bar{\eta}.$$

Since, by (4.2), $\bar{x}^1 \in \beta^1(\bar{p},\bar{\eta})$, using (4.3), one deduces that

$$(4.5)\qquad (\lambda \,\square\, \bar{p}) \bullet_L (\bar{x}^1 - e^1) = \sum_{\xi \in \mathbb{D}} \lambda(\xi)\bar{p}(\xi) \bullet_H (\bar{x}^1(\xi) - e^1(\xi)) \\ \leq \bar{\eta} \bullet_J \bar{z}^1 + \rho(\bar{p},\bar{\eta}) \left(\sum_{\xi\in\mathbb{D}} \lambda(\xi) \right).$$

and, for every $i \neq 1$, since $(\bar{x}^i, \bar{z}^i) \in \beta^i(\bar{p}, \bar{\eta})$, by (4.4),

$$\bar{p} \,\square\, (\bar{x}^i - e^i) \leq W\bar{z}^i + \rho(\bar{p}, \bar{\eta})\mathbb{1}. \tag{4.6}$$

Taking the scalar product with λ and recalling that ${}^tW\lambda = \bar{\eta}$ from the definition of W, we conclude that, for $i \neq 1$,

$$\sum_{\xi \in \mathbb{D}} \lambda(\xi)\bar{p}(\xi) \bullet_H (\bar{x}^i(\xi) - e^i(\xi)) - \rho(\bar{p}, \bar{\eta}) \left(\sum_{\xi \in \mathbb{D}} \lambda(\xi) \right)$$
$$\leq \lambda \bullet_{\mathbb{D}} [W\bar{z}^i] = [{}^tW\lambda] \bullet_J \bar{z}^i = \bar{\eta} \bullet_J \bar{z}^i.$$

Hence, summing over $i \in I$ we have proved the following claim:

Claim 4.2. $(\lambda \square \bar{p}) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i) \leq \bar{\eta} \bullet_J \sum_{i \in I} \bar{z}^i + \#I\big(\sum_{\xi \in \mathbb{D}} \lambda(\xi)\big)\rho(\bar{p}, \bar{\eta})$, *and the equality holds if the equality holds in* (4.5) *and* (4.6).

Claim 4.3. $\sum_{i \in I} \bar{z}^i = 0$ *and* $\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i$.

Proof of Claim 4.3. From Assertion (4.1) (taking successively $p = \bar{p}$ and $\eta = \bar{\eta}$), we get:

$$\bar{\eta} \bullet_J \sum_{i \in I} \bar{z}^i \leq \eta \bullet_J \sum_{i \in I} \bar{z}^i \quad \text{for every } \eta \in \mathbb{R}^J,\ \|\eta\| \leq 1, \tag{4.7}$$

$$(\lambda \,\square\, p) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i) \leq (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i) \tag{4.8}$$
$$\text{for every } p \in \mathbb{R}^L,\ \|\lambda \,\square\, p\| \leq 1.$$

We first prove that $\sum_{i \in I} \bar{z}^i = 0$ by contradiction. Suppose it is not true, from (4.7) we deduce that $\bar{\eta} = -\frac{\sum_{i \in I} \bar{z}^i}{\|\bar{z}^i\|}$. Hence $\|\bar{\eta}\| = 1$, $\rho(\bar{p}, \bar{\eta}) := \max\{0, 1 - \|\lambda \,\square\, \bar{p}\| - \|\bar{\eta}\|\} = 0$ and $\bar{\eta} \bullet_J \sum_{i \in I} \bar{z}^i < 0$. Consequently, from Claim 4.2 one gets:

$$(\lambda \,\square\, \bar{p}) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i) \leq \bar{\eta} \bullet_J \sum_{i \in I} \bar{z}^i + 0 < 0,$$

But, from inequality (4.8), (taking $p = 0$) one gets

$$0 \leq (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i),$$

a contradiction with the above inequality. □

In the same way we now prove the second equality $\sum_{i \in I}(\bar{x}^i - e^i) = 0$ by contradiction. Suppose it is not true, from (4.7) we deduce that $0 < (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i \in I}(\bar{x}^i - e^i)$, $\|\lambda \square \bar{p}\| = 1$ and so $\rho(\bar{p}, \bar{\eta}) := \max\{0, 1 - \|\lambda \square \bar{p}\| - \|\bar{\eta}\|\} = 0$. Consequently, from Claim 4.2, recalling from above that $\sum_{i \in I} \bar{z}^i = 0$ one gets the contradiction:

$$0 < (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i \in I} (\bar{x}^i - e^i) \leq \bar{\eta} \bullet_J \sum_{i \in I} \bar{z}^i + 0 = 0. \qquad \square$$

Claim 4.4. *$\bar{x}^1 \in \beta^1(\bar{p}, \bar{\eta})$ and $\beta^1(\bar{p}, \bar{\eta}) \cap P^1(\bar{x}) = \emptyset$.*

Proof of Claim 4.4. From the fixed-point condition (4.2), $\bar{x}^1 \in \beta^1(\bar{p}, \bar{\eta})$. Now suppose that $\beta^1(\bar{p}, \bar{\eta}) \cap P^1(\bar{x}) \neq \emptyset$ and choose $x^1 \in \beta^1(\bar{p}, \bar{\eta}) \cap P^1(\bar{x})$.

We know that $\alpha^1(\bar{p}, \bar{\eta}) \neq \emptyset$ (see the second step in the proof of Claim 4.1), and we choose $\bar{\bar{x}}^1 \in \alpha^1(\bar{p}, \bar{\eta})$. Suppose first that $\bar{\bar{x}}^1 = x^1$; then, from above $x^1 \in P^1(\bar{x}) \cap \alpha^1(\bar{p}, \bar{\eta})$, which contradicts the fact that this set is empty by Assertion (4.2). Suppose now that $\bar{\bar{x}}^1 \neq x^1$, from Assumption (*C.iii*), $[\bar{\bar{x}}^1, x^1[\cap P^1(\bar{x}) \neq \emptyset$ (recalling that $x^1 \in P^1(\bar{x})$) and clearly $[\bar{\bar{x}}^1, x^1[\subset \alpha^1(\bar{p}, \bar{\eta})$ (since $x^1 \in \beta^1(\bar{p}, \bar{\eta})$ and $\bar{\bar{x}}^1 \in \alpha^1(\bar{p}, \bar{\eta})$). Consequently, $P^1(\bar{x}) \cap \alpha^1(\bar{p}, \bar{\eta}) \neq \emptyset$, which contradicts again Assertion (4.2). $\square$

Claim 4.5. *(a) For every $\xi \in \mathbb{D}$, $\bar{p}(\xi) \neq 0$.*
(b) For all $i \neq 1$, $(\bar{x}^i, \bar{z}^i) \in \beta^i(\bar{p}, \bar{\eta})$ and $\beta^i(\bar{p}, \bar{\eta}) \cap (P^i(\bar{x}) \times Z^i) = \emptyset$.

Proof of Claim 4.5. (a) Indeed, suppose that $\bar{p}(\xi) = 0$, for some $\xi \in \mathbb{D}$. From Claim 4.3, $\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i$, and from the Non-Satiation Assumption at node ξ (for Consumer 1) there exists $x^1 \in P^1(\bar{x})$ such that $x^1(\xi') = \bar{x}^1(\xi')$ for every $\xi' \neq \xi$; from Assertion (4.2), $\bar{x}^1 \in \beta^1(\bar{p}, \bar{\eta})$ and, recalling that $\bar{p}(\xi) = 0$, one deduces that $x^1 \in \beta^1(\bar{p}, \bar{\eta})$. Consequently,

$$\beta^1(\bar{p}, \bar{\eta}) \cap P^1(\bar{x}) \neq \emptyset,$$

which contradicts Claim 4.4.
(b) From the fixed point condition (4.4), for $i \neq 1$ one has $(\bar{x}^i, \bar{z}^i) \in \beta^i(\bar{p}, \bar{\eta})$. Now, suppose that there exists $i \neq 1$ such that $\beta^i(\bar{p}, \bar{\eta}) \cap (P^i(\bar{x}) \times Z^i) \neq \emptyset$ and let $(x^i, z^i) \in \beta^i(\bar{p}, \bar{\eta}) \cap (P^i(\bar{x}) \times Z^i)$. From the Survival Assumption and the fact that $\bar{p}(\xi) \neq 0$ for every $\xi \in \mathbb{D}$ (Part (a)), one deduces that $\alpha^i(\bar{p}, \bar{\eta}) \neq \emptyset$ and we let $(\bar{\bar{x}}^i, \bar{\bar{z}}^i) \in \alpha^i(\bar{p}, \bar{\eta})$.[7]

Suppose first that $\bar{\bar{x}}^i = x^i$, then, from above $(x^i, \bar{\bar{z}}^i) \in [P^i(\bar{x}) \times Z^i] \cap \alpha^i(\bar{p}, \bar{\eta})$, which contradict the fact that this set is empty by Assertion (4.4). Suppose now that $\bar{\bar{x}}^i \neq x^i$, from Assumption (*C.iii*), (recalling that $x^i \in P^i(\bar{x})$) the set $[\bar{\bar{x}}^i, x^i[\cap P^i(\bar{x})$ is nonempty, hence contains a point $x^i(\lambda) := (1-\lambda)\bar{\bar{x}}^i + \lambda x^i$ for some $\lambda \in [0, 1[$. We let $z^i(\lambda) := (1-\lambda)\bar{\bar{z}}^i + \lambda z^i$ and we check that $(x^i(\lambda), z^i(\lambda)) \in \alpha^i(\bar{p}, \bar{\eta})$ (since $(x^i, z^i) \in \beta^i(\bar{p}, \bar{\eta})$ and $(\bar{\bar{x}}^i, \bar{\bar{z}}^i) \in \alpha^i(\bar{p}, \bar{\eta})$). Consequently, $\alpha^i(\bar{p}, \bar{\eta}) \cap (P^i(\bar{x}) \times Z^i) \neq \emptyset$, which contradicts again Assertion (4.4). $\square$

Claim 4.6. $\rho(\bar{p}, \bar{\eta}) = 0$.

[7] Take $\bar{\bar{z}}^i = 0$ and $\bar{\bar{x}}^i = e^i - t\bar{p}$ for $t > 0$ small enough, so that $\bar{\bar{x}}^i \in X^i$ (from the Survival Assumption). Then notice that $\bar{p} \,\square\, (\bar{\bar{x}}^i - e^i) = -t(\bar{p} \,\square\, \bar{p}) \ll 0 \leq 0 + \rho(\bar{p}, \bar{\eta})\mathbb{1}$.

Proof of Claim 4.6. We first prove that the budget constraints of consumers $i \in I$, $i \neq 1$, are binded, that is:

$$\bar{p} \,\square\, (\bar{x}^i - e^i) = W\bar{z}^i + \rho(\bar{p},\bar{\eta})\mathbb{1}, \quad \text{for every} \quad i \neq 1. \tag{4.9}$$

Indeed, if it is not true, there exist $i \in I$, $i \neq 1$ such that

$$\bar{p} \,\square\, (\bar{x}^i - e^i) \leq W\bar{z}^i + \rho(\bar{p},\bar{\eta})\mathbb{1},$$

with a strict inequality for some component $\xi \in \mathbb{D}$. But $\sum_{i\in I} \bar{x}^i = \sum_{i\in I} e^i$ (Claim 4.3) and from the Non-Satiation Assumption at node ξ (for consumer i), there exists $x^i \in P^i(\bar{x})$ such that $x^i(\xi') = \bar{x}^i(\xi')$ for every $\xi' \neq \xi$. Consequently, we can choose $x \in [x^i, \bar{x}^i[$ close enough to $\bar{x}^i$ so that $(x, \bar{z}^i) \in \beta^i(\bar{p},\bar{\eta})$. But, from the Local Non-Satiation (Assumption (K.ii)), $[x^i, \bar{x}^i[\subset P^i(\bar{x})$. Consequently, $\beta^i(\bar{p},\bar{\eta}) \cap (P^i(\bar{x}) \times Z^i) \neq \emptyset$ which contradicts Claim 4.5.

In the same way, we prove that the budget constraint of Consumer 1 is binded. Consequently, from Claim 4.2, using the facts that $\sum_{i\in I}(\bar{x}^i - e^i) = 0$ and $\sum_{i\in I} \bar{z}^i = 0$ (by Claim 4.3) one has

$$0 = (\lambda \,\square\, \bar{p}) \bullet_L \sum_{i\in I}(\bar{x}^i - e^i) - \bar{\eta} \bullet_J \sum_{i\in I} \bar{z}^i = \#I \left(\sum_{\xi\in\mathbb{D}} \lambda(\xi) \right) \rho(\bar{p},\bar{\eta}).$$

Since $\sum_{\xi\in\mathbb{D}} \lambda(\xi) > 0$, we conclude that $\rho(\bar{p},\bar{\eta}) = 0$. □

Claim 4.7. *For every* $i \in I$, $(\bar{x}^i, \bar{z}^i) \in B^i_{\mathcal{F}}(\bar{p},\bar{q})$ *and* $[P^i(\bar{x}) \times Z^i] \cap B^i_{\mathcal{F}}(\bar{p},\bar{q}) = \emptyset$.

Proof of Claim 4.7. Since $\rho(\bar{p},\bar{\eta}) = 0$ (From Claim 4.6), for every $i \neq 1$, $B^i_{\mathcal{F}}(\bar{p},\bar{q}) = \beta^i(\bar{p},\bar{q})$. Hence, from Claim 4.5 we deduce that Claim 4.7 is true for every consumer $i \neq 1$.

About the first consumer, we first notice that $B^1_{\mathcal{F}}(\bar{p},\bar{q}) \subset \beta^1(\bar{p},\bar{\eta}) \times Z^1$. So, in view of Claim 4.5, the proof will be complete if we show that $(\bar{x}^1, \bar{z}^1) \in B^1_{\mathcal{F}}(\bar{p},\bar{q})$. But since the budget constraints of agent $i \in I$, $i \neq 1$, are binded (see the proof of Claim 4.6), $\sum_{i\in I}(\bar{x}^i - e^i) = 0$ and $\sum_{i\in I} \bar{z}^i = 0$ (Claim 4.3), we conclude that

$$\bar{p} \,\square\, (\bar{x}^1 - e^1) = -\sum_{i\neq 1} \bar{p} \,\square\, (\bar{x}^i - e^i) = -\sum_{i\neq 1} W\bar{z}^i = W\bar{z}^1,$$

which ends the proof of the claim. □

4.2 Proof in the general case

We now give the proof of Theorem 3.1, without considering the additional Assumption (K), as in the previous section. We will first enlarge the strict preferred sets as in Gale-Mas Colell, and then truncate the economy $\mathcal{E}$ by a standard argument to define a new economy $\hat{\mathcal{E}}_r$, which satisfies all the assumptions of $\mathcal{E}$, together with the additional Assumption (K). From the previous section, there exists an equilibrium of $\hat{\mathcal{E}}_r$ and we will then check that it is also an equilibrium of $\mathcal{E}$.

4.2.1 Enlarging the preferences as in Gale-Mas Colell

The original preferences P^i are replaced by the "enlarged" ones $\hat{P}^i$ defined as follows. For every $i \in I$, $\bar{x} \in \prod_{i \in I} X^i$ we let

$$\hat{P}^i(\bar{x}) := \bigcup_{x^i \in P^i(\bar{x})}]\bar{x}^i, x^i] = \left\{ \bar{x}^i + t(x^i - \bar{x}^i) \mid t \in]0,1],\ x^i \in P^i(\bar{x}) \right\}.$$

The next proposition shows that $\hat{P}^i$ satisfies the same properties as P^i, for every $i \in I$, together with the additional Local Non-Satiation Assumption (K.ii).

Proposition 4.1. *Under* (C), *for every $i \in I$ and every $\bar{x} \in \prod_{i \in I} X^i$ one has:*

(i) $P^i(\bar{x}) \subset \hat{P}^i(\bar{x}) \subset X^i$;

(ii) the correspondence $\hat{P}^i$ is lower semicontinuous at $\bar{x}$ and $\hat{P}^i(\bar{x})$ is convex;

(iii) for every $y^i \in \hat{P}^i(\bar{x})$ for every $(x')^i \in X^i$, $(x')^i \neq y^i$ then $[(x')^i, y^i[\cap \hat{P}^i(\bar{x}) \neq \emptyset$;

(iv) $\bar{x}^i \notin \hat{P}^i(\bar{x})$;

(v) (Non-Satiation at Every Node) if $\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i$, for every $\xi \in \mathbb{D}$, there exists $x \in \prod_{i \in I} X^i$ such that, for each $\xi' \neq \xi$, $x^i(\xi') = \bar{x}^i(\xi')$ and $x^i \in \hat{P}^i(\bar{x})$;

(vi) for every $y^i \in \hat{P}^i(\bar{x})$, then $[y^i, \bar{x}^i[\subset \hat{P}^i(\bar{x})$.

Proof. Let $\bar{x} \in \prod_{i \in I} X^i$ and let $i \in I$.

Part (i). It follows by the convexity of X^i, for every $i \in I$.

Part (ii). Let $y^i \in \hat{P}^i(\bar{x})$ and consider a sequence $(\bar{x}_n)_n \subset \prod_{i \in I} X^i$ converging to $\bar{x}$. Since $y^i \in \hat{P}^i(\bar{x})$, then $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$ for some $x^i \in P^i(\bar{x})$ and some $t \in]0,1]$. Since P^i is lower semicontinuous, there exists a sequence (x_n^i) converging to x^i such that $x_n^i \in P^i(\bar{x}_n)$ for every $n \in \mathbb{N}$. Now define $y_n^i := \bar{x}_n^i + t(x_n^i - \bar{x}_n^i) \in]\bar{x}_n^i, x_n^i]$: then $y_n^i \in \hat{P}^i(\bar{x}_n)$ and obviously the sequence (y_n^i) converges to y^i. This shows that $\hat{P}^i$ is lower semicontinuous at $\bar{x}$.

To show that $\hat{P}^i(\bar{x})$ is convex, let $y_1^i, y_2^i \in \hat{P}^i(\bar{x})$, let $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, such that $\lambda_1 + \lambda_2 = 1$. Then $y_k^i = \bar{x}^i + t_k(x_k^i - \bar{x}^i)$ for some $t_k \in]0,1]$ and some $x_k^i \in P^i(\bar{x})$ $(k = 1, 2)$. One has

$$\lambda_1 y_1^i + \lambda_2 y_2^i = \bar{x}^i + (\lambda_1 t_1 + \lambda_2 t_2)(x^i - \bar{x}^i),$$

where $x^i := (\lambda_1 t_1 x_1^i + \lambda_2 t_2 x_2^i)/(\lambda_1 t_1 + \lambda_2 t_2) \in P^i(\bar{x})$ (since $P^i(\bar{x})$ is convex, by Assumption (C)) and $\lambda_1 t_1 + \lambda_2 t_2 \in]0,1]$. Hence $\lambda_1 y_1^i + \lambda_2 y_2^i \in \hat{P}^i(\bar{x})$.

Part (iii). Let $y^i \in \hat{P}^i(\bar{x})$ and let $(x')^i \in X^i$, $(x')^i \neq y^i$. From the definition of $\hat{P}^i$, $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$ for some $x^i \in P^i(\bar{x})$ and some $t \in]0,1]$. Suppose first that $x^i = (x')^i$, then $y^i \in]\bar{x}^i, x^i[\subset \hat{P}^i(\bar{x})$. Consequently, $[(x')^i, y^i[\cap \hat{P}^i(\bar{x}) \neq \emptyset$. Suppose now that $x^i \neq (x')^i$; since P^i satisfies Assumption (*C.iii*), there exists $\lambda \in [0,1[$ such that $x^i(\lambda) = (x')^i + \lambda(x^i - (x')^i) \in P^i(\bar{x})$. We let

$$z := [\lambda(1-t)\bar{x}^i + t(1-\lambda)(x')^i + t\lambda x^i]/\alpha \text{ with } \alpha := t + \lambda(1-t),$$

and we check that $z = [\lambda(1-t)\bar{x}^i + tx^i(\lambda)]/\alpha \in]\bar{x}^i, x^i(\lambda)]$, with $x^i(\lambda) \in P^i(\bar{x})$, hence $z \in \hat{P}^i(\bar{x})$. Moreover, $z := [\lambda y^i + t(1-\lambda)(x')^i]/\alpha \in [(x')^i, y^i[$. Consequently, $[(x')^i, y^i[\cap \hat{P}^i(\bar{x}) \neq \emptyset$, which ends the proof of (*iii*).
Parts (iv), (v) and (vi). They follow immediately by the definition of $\hat{P}^i$ and the properties satisfied by P^i in (C). □

4.2.2 Truncating the economy

We now define the "truncated economy" as follows.

For every $i \in I$, $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, we let $\hat{X}^i(\lambda)$ and $\hat{Z}^i(\lambda)$ be the projections of $B(\lambda)$ on X^i and Z^i, respectively, namely

$$\hat{X}^i(\lambda) := \left\{ x^i \in X^i : \exists (x^j)_{j \neq i} \in \prod_{j \neq i} X^j,\ \exists z \in \prod_{i \in I} Z^i,\ (x,z) \in B(\lambda) \right\}$$

and

$$\hat{Z}^i(\lambda) := \left\{ z^i \in Z^i : \exists (z^j)_{j \neq i} \in \prod_{j \neq i} Z^j,\ \exists x \in \prod_{i \in I} X^i,\ (x,z) \in B(\lambda) \right\}.$$

By Assumption (B_λ), the set $B(\lambda)$ is bounded, hence the sets $\hat{X}^i(\lambda)$ and $\hat{Z}^i(\lambda)$ are also bounded subsets of $\mathbb{R}^L$ and $\mathbb{R}^J$, respectively. So there exists a real number $r > 0$ such that, for every agent $i \in I$, $\hat{X}^i(\lambda) \subset \operatorname{int} B_L(0,r)$ and $\hat{Z}^i(\lambda) \subset \operatorname{int} B_J(0,r)$. The truncated economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ is the collection

$$(\hat{\mathcal{E}}_r, \mathcal{F}_r) = \left[\mathbb{D}, H, I, (X^i_r, \hat{P}^i_r, e^i)_{i \in I}; J, (Z^i_r)_{i \in I}, (\xi(j))_{j \in J}, V\right],$$

where, for every $x \in \prod_{i \in I} X^i$,

$$X^i_r = X^i \cap B_L(0,r),\ Z^i_r = Z^i \cap B_J(0,r) \text{ and } \hat{P}^i_r(x) = \hat{P}^i(x) \cap \operatorname{int} B_L(0,r).$$

The existence of equilibria of $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ is then a consequence of Section 4.1, that is, Theorem 3.1 with the additional Assumption (K). We just have to check that Assumption (K) and all the assumptions of Theorem 3.1 are satisfied by $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$. In view of Proposition 4.1, this is clearly the case for all the assumptions but the Survival Assumptions (*C.vi*) and (*F.iii*), that are proved via a standard argument (that we recall hereafter).

Indeed we first notice that $(e^i, 0)_{i \in I}$ belongs to $B(\lambda)$, hence, for every $i \in I$, $e^i \in \hat{X}^i(\lambda) \subset \operatorname{int} B_L(0,r)$. Recalling that $e^i \in \operatorname{int} X^i$ (from the Survival Assumption), we deduce that $e^i \in \operatorname{int} X^i \cap \operatorname{int} B_L(0,r) \subset \operatorname{int}[X^i \cap B_L(0,r)] = \operatorname{int} X^i_r$. Similarly, for every $i \in I$, $0 \in \hat{Z}^i(\lambda) \subset \operatorname{int} B_J(0,r)$. Consequently $0 \in Z^i_r = Z^i \cap B_J(0,r)$. Moreover, for some $i_0 \in I$ one has $0 \in \operatorname{int} Z^{i_0}$ (by Assumption (*F.iii*)), and, as above, $0 \in \operatorname{int} B_J(0,r)$. Consequently, $0 \in \operatorname{int}[Z^{i_0} \cap B_J(0,r)] = \operatorname{int} Z^{i_0}_r$.

The end of the proof of Theorem 3.1 consists to show that equilibria of $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ are in fact also equilibria of $(\mathcal{E}, \mathcal{F})$, which thus exist from above.

Proposition 4.2. *Under Assumption (B_λ), if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ such that $\bar{p} \in B_L(0,1)$ and ${}^tW\lambda \in N_{Z^1_r \cap B_J(0,1)}(\bar{z}^1)$, then it is also an equilibrium of $(\mathcal{E}, \mathcal{F})$ and ${}^tW\lambda \in N_{Z^1}(\bar{z}^1)$.*

Proof. Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an equilibrium of the economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$. In view of the definition of an equilibrium, to prove that it is also an equilibrium of $(\mathcal{E}, \mathcal{F})$ we only have to check that $[P^i(\bar{x}) \times Z^i] \cap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset$ for every $i \in I$, where $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$ denotes the budget set of agent i in the economy $(\mathcal{E}, \mathcal{F})$.

Assume, on the contrary, that, for some $i \in I$ the set $[P^i(\bar{x}) \times Z^i] \cap B^i_{\mathcal{F}}(\bar{p}, \bar{q})$ is nonempty, hence contains a couple (x^i, z^i). Clearly the allocation $(\bar{x}, \bar{z})$ belongs to the set $B(\lambda)$, hence for every $i \in I$, $\bar{x}^i \in \hat{X}^i(\lambda) \subset \operatorname{int} B_L(0,r)$ and $\bar{z}^i \in \hat{Z}^i(\lambda) \subset \operatorname{int} B_J(0,r)$. Thus, for $t \in]0,1]$ sufficiently small, $x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i) \in \operatorname{int} B_L(0,r)$ and $z^i(t) := \bar{z}^i + t(z^i - \bar{z}^i) \in \operatorname{int} B_J(0,r)$. Clearly $(x^i(t), z^i(t))$ belongs to the budget set $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$ of agent i (for the economy $(\mathcal{E}, \mathcal{F})$) and since $x^i(t) \in X^i_r := X^i \cap B_L(0,r)$, $z^i(t) \in Z^i_r := Z^i \cap B_J(0,r)$, the couple $(x^i(t), z^i(t))$ belongs also to the budget set $B^i_r(\bar{p}, \bar{q})$ of agent i (in the economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$). From the definition of $\hat{P}^i$, we deduce that $x^i(t) \in \hat{P}^i(\bar{x})$ (since from above $x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i)$ and $x^i \in P^i(\bar{x})$), hence $x^i(t) \in \hat{P}^i_r(\bar{x}) := \hat{P}^i(\bar{x}) \cap \operatorname{int} B_L(0,r)$. We have thus shown that, for $t \in]0,1]$ small enough, $(x^i(t), z^i(t)) \in [\hat{P}^i_r(\bar{x}) \times Z^i_r] \cap B^i_r(\bar{p}, \bar{q})$. This contradicts the fact that this set is empty, since $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$.

We now prove that $\bar{\eta} := {}^tW_{\mathcal{F}}(\bar{p}, \bar{q})\lambda \in N_{Z^1}(\bar{z}^1)$. We let $z^1 \in Z^1$ and we show that $\bar{y} \bullet_J \bar{z}^1 \geq \bar{\eta} \bullet_J z^1$. We have seen above that $\bar{z}^1 \in \hat{Z}^1(\lambda) \subset \operatorname{int} B_J(0,r)$. Then, for $t > 0$ small enough, $z(t) := \bar{z}^1 + t(z^1 - \bar{z}^1) \in \operatorname{int} B_J(0,r)$ and $z(t) \in Z^1$, by the convexity of Z^1. Consequently, for t small enough, $z(t) \in Z^1_r = Z^1 \cap B_J(0,r)$ and using the fact that $\bar{\eta} \in N_{Z^1_r}(\bar{z}^1)$, we deduce that

$$\bar{\eta} \bullet_J \bar{z}^1 \geq \bar{\eta} \bullet_J z(t) = \bar{\eta} \bullet_J \bar{z}^1 + t\bar{\eta} \bullet_J (z^1 - \bar{z}^1),$$

hence $\bar{\eta} \bullet_J z^1 \leq \bar{\eta} \bullet_J \bar{z}^1$. □

5. Appendix

5.1 Retrading financial assets and equilibria

In this section we will show that, if every asset of the financial structure $\mathcal{F}$ can be retraded at each node, the previous equilibrium notion coincides with another concept widely used in the literature (see for example Magill-Quinzii [23]).

To every asset $j \in J$ and every node $\xi' > \xi(j)$ which is not a *maturity node*[8] of j we define the new asset $\widetilde{\jmath} = (j, \xi')$, which is issued at ξ', and has the same payoffs as asset j at every node which succeeds ξ'. For the sake of convenient notations, we shall allow to retrade every asset j at every node $\xi' \in \mathbb{D}$.[9]

Throughout this section we shall assume that the portfolios are unconstrained, that is, $Z^i = \mathbb{R}^J$, for every $i \in I$.

Definition 5.1. *The retrading of asset $j \in J$ at node $\xi' \in \mathbb{D}$, denoted $\widetilde{\jmath} = (j, \xi')$, is the asset issued at ξ', that is, $\xi(j, \xi') = \xi'$, and whose flow of payoffs is given by*
$\widetilde{v}(p, \xi, (j, \xi')) = v(p, \xi, j)$, if $\xi' < \xi$;
$\widetilde{v}(p, \xi, (j, \xi')) = 0$, otherwise.

Given the financial structure $\mathcal{F} = (J, (\xi(j))_{j \in J}, V)$, we associate a new financial structure $\widetilde{\mathcal{F}} = (\widetilde{J}, (\xi(\widetilde{\jmath}))_{\widetilde{\jmath} \in \widetilde{J}}, \widetilde{V})$, called the retrading extension *of $\mathcal{F}$, which consists of all the retradings (j, ξ') of asset $j \in J$ at node $\xi' \in \mathbb{D}$. Hence $\widetilde{J} = J \times \mathbb{D}$ and the $\mathbb{D} \times \widetilde{J}$-matrix $\widetilde{V}(p)$ has for coefficients $\widetilde{v}(p, \xi, (j, \xi'))$, as defined above.*

We denote by $q_j(\xi')$ the price of asset (j, ξ') (i.e., the retrading of asset j at node ξ'), which is sometimes also called the retrading price of asset j at node ξ'. So, for the financial structure $\widetilde{\mathcal{F}}$, both the asset price vector $q = (q_j(\xi'))_{j \in J, \xi' \in \mathbb{D}}$ and the portfolio $z = (z_j(\xi'))_{j \in J, \xi' \in \mathbb{D}}$ now belong to $\mathbb{R}^{J \times \mathbb{D}}$. Given $p \in \mathbb{R}^L$, q and z in $\mathbb{R}^{J \times \mathbb{D}}$, the full financial return of $\widetilde{\mathcal{F}}$ at node $\xi \in \mathbb{D}$ is

[8] We recall that the maturity nodes of an asset j are the nodes $\xi > \xi(j)$ such that $v(p, \xi, j) \neq 0$ and $v(p, \xi', j) = 0$ for every $\xi' > \xi$.

[9] In particular, if ξ' is a terminal node ($\xi' \in \mathbb{D}_T$) the payoff of the asset (j, ξ') is zero (i.e., $\widetilde{v}(p, \xi, (j, \xi')) = 0$ for every $\xi \in \mathbb{D}$). However, these assets do not affect the equilibrium notion since, under the Non-Satiation Assumption at every Node, the corresponding equilibrium price $\bar{q}_{(j, \xi')}$ must be zero (otherwise it would lead to an arbitrage situation which does not prevail at equilibrium).

$$\begin{aligned}
&[W_{\widetilde{\mathcal{F}}}(p,q)z](\xi)\\
&= \sum_{(j,\xi')\in J\times\mathbb{D}} \widetilde{v}(p,\xi,(j,\xi'))z_j(\xi') - \sum_{(j,\xi')\in J\times\mathbb{D}} \delta_{\xi,\xi(j,\xi')}q_j(\xi')z_j(\xi')\\
&= \sum_{j\in J}\sum_{\xi'<\xi} v(p,\xi,j)z_j(\xi') - \sum_{j\in J} q_j(\xi)z_j(\xi).
\end{aligned}$$

We now give the definition of equilibrium which is most often used when retrading is allowed. Given the financial structure $\mathcal{F} = (J,(\xi(j))_{j\in J},V)$ and given $p\in\mathbb{R}^L$, $q\in\mathbb{R}^{J\times\mathbb{D}}$, we first define the budget set:

$$\widetilde{B}^i_{\mathcal{F}}(p,q) = \left\{(x^i,y^i)\in X^i\times\mathbb{R}^{J\times\mathbb{D}} : p\,\square\,(x^i-e^i)\le \widetilde{W}_{\mathcal{F}}(p,q)y^i\right\}$$

where we let, for $y=(y_j(\xi))_{(j,\xi)\in J\times\mathbb{D}}\in\mathbb{R}^{J\times\mathbb{D}}$:

$$\begin{aligned}
&\left[\widetilde{W}_{\mathcal{F}}(p,q)y\right](\xi)\\
&=\begin{cases} -\sum_{j\in J} q_j(\xi_0)y_j(\xi_0), & \xi=\xi_0,\\ \sum_{j\in J} v(p,\xi,j)y_j(\xi^-) + \sum_{j\in J} q_j(\xi)y_j(\xi^-) - \sum_{j\in J} q_j(\xi)y_j(\xi), & \forall\xi\neq\xi_0.\end{cases}
\end{aligned}$$

We recall that we have allowed the retrading of assets at terminal nodes, for the sake of convenient notations; so we don't need above to distinguish the cases $\xi\in\mathbb{D}_T$ and $\xi\notin\mathbb{D}_T$.[10]

Definition 5.2. *An* equilibrium with retrading *of the economy $\mathcal{E}$ and the financial structure $\mathcal{F} = (J,(\xi(j))_{j\in J},V)$ is a collection of strategies and prices $(\bar{x},\bar{y},\bar{p},\bar{q})\in(\mathbb{R}^L)^I\times(\mathbb{R}^{J\times\mathbb{D}})^I\times\mathbb{R}^L\setminus\{0\}\times\mathbb{R}^{J\times\mathbb{D}}$ such that*
(a) for every $i\in I$, $(\bar{x}^i,\bar{y}^i)\in\widetilde{B}^i_{\mathcal{F}}(\bar{p},\bar{q})$ and $\left[P^i(\bar{x})\times\mathbb{R}^{J\times\mathbb{D}}\right]\cap\widetilde{B}^i_{\mathcal{F}}(\bar{p},\bar{q})=\emptyset$;
(b) $\sum_{i\in I}\bar{x}^i=\sum_{i\in I}e^i$ and $\sum_{i\in I}\bar{y}^i=0$.

The next proposition shows that, for a given exchange economy $\mathcal{E}$, equilibria with retrading associated to the financial structure $\mathcal{F}$ are in a one-to-one correspondence with the equilibria associated to the retrading extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$. The correspondence will only change the equilibrium portfolios via the mapping $\varphi\colon\ \mathbb{R}^{J\times\mathbb{D}}\longrightarrow\mathbb{R}^{J\times\mathbb{D}}$ defined by

$$\varphi(z)(j,\xi)=\sum_{\xi'\le\xi} z(j,\xi'),\quad\text{for every}\quad z\in\mathbb{R}^{J\times\mathbb{D}}.$$

[10] But again, at equilibrium, under a standard Non-Satiation assumption (see assumption NS), a no-arbitrage argument will imply that $q_j(\xi)=0$ if $\xi\in\mathbb{D}_T$. So allowing assets to be emitted at terminal nodes does not matter.

and φ is easily shown to be linear and bijective.[11]

Proposition 5.1. *Let $\mathcal{E}$ be an exchange economy, let $\mathcal{F} = (J, (\xi(j))_{j\in J}, V)$ and let $\widetilde{\mathcal{F}} = (\widetilde{J}, (\xi(j))_{j\in \widetilde{J}}, \widetilde{V})$ be the retrading extension of $\mathcal{F}$. Then the two following conditions are equivalent:*
(i) $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}, \widetilde{\mathcal{F}})$;
(ii) $(\bar{x}, (\varphi(\bar{z}^i))_{i\in I}, \bar{p}, \bar{q})$ is an equilibrium with retrading of $(\mathcal{E}, \mathcal{F})$.

Proof. Since φ is linear and bijective, the equality $\sum_{i\in I} \bar{z}^i = 0$ holds if and only if $\sum_{i\in I} \varphi(\bar{z}^i) = 0$. Thus the end of the proof is a consequence of the following claim.

Claim 5.1. *For every $(p, q) \in \mathbb{R}^L \times \mathbb{R}^{J\times\mathbb{D}}$ one has*
(i) for every $z \in \mathbb{R}^{J\times\mathbb{D}}$, $W_{\widetilde{\mathcal{F}}}(p, q)z = \widetilde{W}_{\mathcal{F}}(p, q)\varphi(z)$;
(ii) $B^i_{\widetilde{\mathcal{F}}}(p, q) = \{(x^i, z^i) \mid (x^i, \varphi(z^i)) \in \widetilde{B}^i_{\mathcal{F}}(p, q)\}$.

Proof. Part (i). For $\xi = \xi_0$, we have $\varphi(z)(j, \xi_0) = z(j, \xi_0)$ for every $j \in J$; from the definitions of $W_{\widetilde{\mathcal{F}}}(p, q)$ and $\widetilde{W}_{\mathcal{F}}(p, q)$, we get:

$$\begin{aligned}[\widetilde{W}_{\mathcal{F}}(p, q)\varphi(z)](\xi_0) &= -\sum_{j\in J} q_j(\xi_0)[\varphi(z)(j, \xi_0)] \\ &= -\sum_{j\in J} q_j(\xi_0) z_j(\xi_0) = [W_{\widetilde{\mathcal{F}}}(p, q)z](\xi_0).\end{aligned}$$

For $\xi \neq \xi_0$ we have

$$\begin{aligned}&[\widetilde{W}_{\mathcal{F}}(p, q)\varphi(z)](\xi) \\ &\quad= \sum_{j\in J} v(p, \xi, j)\varphi(z)(j, \xi^-) + \sum_{j\in J} q_j(\xi)\varphi(z)(j, \xi^-) - \sum_{j\in J} q_j(\xi)\varphi(z)(j, \xi) \\ &\quad= \sum_{j\in J} v(p, \xi, j) \sum_{\xi'\leq\xi^-} z_j(\xi') - \sum_{j\in J} q_j(\xi)[\varphi(z)(j, \xi) - \varphi(z)(j, \xi^-)] \\ &\quad= \sum_{(j,\xi')\in J\times\mathbb{D}} \widetilde{v}(p, \xi, (j, \xi'))z_j(\xi') - \sum_{j\in J} q_j(\xi)z_j(\xi) \\ &\quad= [W_{\widetilde{\mathcal{F}}}(p, q)z](\xi).\end{aligned}$$

Part (ii). It is a direct consequence of *(i)*. □

[11] It is easy to see that the inverse of φ is the mapping $\psi\colon \mathbb{R}^{J\times\mathbb{D}} \longrightarrow \mathbb{R}^{J\times\mathbb{D}}$ defined by $\psi(z)(j, \xi) = z(j, \xi) - z(j, \xi^-)$, if $\xi \neq \xi_0$, and $\psi(z)(j, \xi_0) = z(j, \xi_0)$, if $\xi = \xi_0$.

5.2 Relationship between $rank\ V_{\mathcal{F}}$ and $rank\ W_{\mathcal{F}}$ in a multi-period model

The next Proposition shows that several properties of the two-date model also hold in the case of short-lived financial structures. First, the list of emission nodes $(\xi(j))_{j\in J}$ of the (non-zero) short-lived assets is uniquely determined by the knowledge of the return matrix $V_{\mathcal{F}}(p)$, and, secondly, the relationship between the ranks of the matrices $V_{\mathcal{F}}(p)$ and $W_{\mathcal{F}}(p,q)$ can be simply formulated.

Proposition 5.2. *For short-lived financial structures $\mathcal{F}$, the following holds:*
(a) if, for every $j \in J$, $V_{\mathcal{F}}(p,j) \neq 0$, then the emission node $\xi(j)$ is uniquely determined by the knowledge of the payoff vector $V_{\mathcal{F}}(p,j)$, that is, $\xi(j) = \xi^-$ for every $\xi \in \mathbb{D}$ such that $v(p,\xi,j) \neq 0$;
(b) $rank\ V_{\mathcal{F}}(p) \leq rank\ W_{\mathcal{F}}(p,q)$ for every $(p,q) \in \mathbb{R}^L \times \mathbb{R}^J$;
(c) $rank\ V_{\mathcal{F}}(p) = rank\ W_{\mathcal{F}}(p,q)$ if ${}^tW_{\mathcal{F}}(p,q)\lambda = 0$ for some $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$.

In the following, we omit the subscript $\mathcal{F}$ of the matrices $V_{\mathcal{F}}(p)$ and $W_{\mathcal{F}}(p)$.

Proof. Part (a) is straightforward. We prepare the proofs of Part (b) and (c) by introducing some notations and definitions. We let, for $t = 1, \ldots, T+1$, the set $J_t = \{j \in J \mid \xi(j) \in \mathbb{D}_{t-1}\}$.

We give the proof under the additional assumption that $J_t \neq \emptyset$ for $t \in [1,T]$ and $J_{T+1} = \emptyset$ (and we let the reader adapt this proof to the general case). Then the sets J_t $(t \in [1,T])$ define a partition of the set J and we write every $z \in \mathbb{R}^J$ as $z = (z_t)$ with $z_t \in \mathbb{R}^{J_t}$. We also define the $\mathbb{D}_t \times J_\tau$ sub-matrix $V_{t,\tau}(p)$ of $V(p)$ and the $\mathbb{D}_t \times J_\tau$ sub-matrix $W_{t,\tau}(p,q)$ of $W(p,q)$, for $t \in \mathcal{T}$ and $\tau = 1, \ldots, T$.

In this case, the matrices $V(p)$ and $W(p,q)$ can be written as follows:

$$
\begin{array}{c} \\ V(p) = \end{array}
\begin{array}{c}
\begin{array}{ccccc} J_1 & J_2 & \ldots & J_{T-1} & J_T \end{array} \\
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
V_{1,1}(p) & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & V_{T-1,T-1}(p) & 0 \\
0 & 0 & \ldots & 0 & V_{T,T}(p)
\end{pmatrix}
\end{array}
\begin{array}{c} \mathbb{D}_0 \\ \mathbb{D}_1 \\ \ldots \\ \mathbb{D}_{T-1} \\ \mathbb{D}_T \end{array},
$$

$$
W(p,q) = \begin{pmatrix}
W_{0,1}(p,q) & 0 & \ldots & 0 & 0 \\
V_{1,1}(p) & W_{1,2}(p,q) & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & V_{T-1,T-1}(p) & W_{T-1,T}(p,q) \\
0 & 0 & \ldots & 0 & V_{T,T}(p)
\end{pmatrix}
\begin{array}{c} \mathbb{D}_0 \\ \mathbb{D}_1 \\ \ldots \\ \mathbb{D}_{T-1} \\ \mathbb{D}_T \end{array}.
$$

To see the above, it suffices to check that, for every (p, q), one has $V_{0,\tau}(p) = 0$ for every τ, $V_{t,\tau}(p) = 0$ if $t \neq \tau$, $W_{0,\tau}(p, q) = 0$, for every $\tau \neq 1$, $W_{t,\tau}(p, q) = 0$ if $\tau \geq t + 2$ and $W_{t,t}(p, q) = V_{t,t}(p)$ for every $t \geq 1$.
Part (b). We first prove it under the additional assumption that $rank\, V(p) = \#J$ (i.e., $V(p)$ is one-to-one). Let $z = (z_t) \in \prod_t \mathbb{R}^{J_t}$ be such that $W(p, q)z = 0$; then one has

$$V_{1,1}(p)z_1 + W_{1,2}(p, q)z_2 = 0,$$
$$\dots$$
$$V_{T-1,T-1}(p)z_{T-1} + W_{T-1,T}(p, q)z_T = 0,$$
$$V_{T,T}(p)z_T = 0.$$

One notices that $rank\, V(p) = \sum_{t=1}^{T} rank\, V_{t,t}(p)$. So, for every t, $rank\, V_{t,t}(p) = \sharp J_t$ (hence $rank\, V(p) = \sharp J$) and each matrix $V_{t,t}(p)$ is one-to-one. From above, by an easy backward induction argument, we deduce that $z_T = 0$, then $z_{T-1} = 0, \dots, z_1 = 0$. Thus $z = 0$ and we have proved that $W(p, q)$ is also one-to-one, that is, $rank\, W(p, q) = \sharp J$.

Suppose now that $rank\, V(p) < \#J$. By eliminating columns of the matrix $V(p)$ we can consider a set $\widetilde{J} \subset J$ and a $(\mathbb{D} \times \widetilde{J})$-sub-matrix $\widetilde{V}(p)$ of $V(p)$ such that $rank\, V(p) = \#\widetilde{J} = rank\, \widetilde{V}(p)$ and the matrix $\widetilde{W}(p, q)$ is defined in a similar way. From the first part of the proof of Part (b), $rank\, \widetilde{V}(p) \leq rank\, \widetilde{W}(p, q)$, and clearly $rank\, \widetilde{W}(p, q) \leq rank\, W(p, q)$. Hence $rank\, V(p) \leq rank\, W(p, q)$.
Part (c). We denote by $V(p, \xi)$ and $W(p, q, \xi)$, respectively, the rows of the matrices $V(p)$ and $W(p, q)$. Since ${}^tW(p, q)\lambda = 0$, from Theorem 2.1 we get

$$\lambda(\xi(j))q_j = \sum_{\xi' \in \xi(j)^+} \lambda(\xi')v(p, \xi', j), \quad \text{for every } j \in J.$$

Consequently, we have:
for $\xi \in \mathbb{D}_T$, $W(p, q, \xi) = V(p, \xi)$ and
for $\xi \notin \mathbb{D}_T$, $W(p, q, \xi) + [1/\lambda(\xi)] \sum_{\xi' \in \xi^+} \lambda(\xi')V(p, \xi') = V(p, \xi)$ (recalling that $V(p, \xi_0) = 0$).

Hence, for every $\xi \in \mathbb{D}$, $W(p, q, \xi)$ belongs to the vector space spanned by the vectors $\{V(p, \xi) \mid \xi \in \mathbb{D}\}$, thus we conclude that $rank\, W(p, q) \leq rank\, V(p)$. □

Remark 5.1 (Long-lived assets). The inequality $rank\, V(p) \leq rank\, W(p, q)$ (Assertion (b) of Proposition 5.2) may not be true in the case of long-lived assets. Consider a stochastic economy with $T = 2$ and three nodes, namely $\mathbb{D} = \{0, 1, 2\}$, and two assets j_1, j_2, where j_1 is emitted at node 0 and pays -1 a node 1, 1 at node 2, j_2 is emitted at node 1 and gives 1 at node 2. Consider the asset price $q = (0, 1)$; then the matrices of returns are

$$V = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 1 \end{pmatrix},$$

and $rank\ W(q) = 1 < rank\ V = 2$. □

Assertion (a) of Proposition 5.2 may not be true in the case of long-lived assets, that is, the payoff matrix may not suffice to describe the financial structure. Consider the above example: then V is also the return matrix of the financial structure $\mathcal{F}'$ consisting of two assets $\{j_1, j_2'\}$, where j_1 is defined as previously and j_2' has for emission node 0 and pays 1 at node 2. It is clear, however, that, for $q = (0, 1)$, the full matrix of returns $W_{\mathcal{F}'}(q)$ is different from $W_{\mathcal{F}}(q)$. □

5.3 Proof of Proposition 3.1 on the Boundedness Assumption B_λ

We will use the following lemma.

Lemma 5.1. *Let A be a compact subset of $\mathbb{R}^n$ and let $W(\alpha)\colon\ \mathbb{R}^J \longrightarrow \mathbb{R}^{\mathbb{D}}$ ($\alpha \in A$) be a linear mapping such that the application $\alpha \mapsto W(\alpha)$ is continuous and rank $W(\alpha) = \sharp J$. Then there exists $c > 0$ such that:*

$$\|W(\alpha)z\| \geq c\|z\| \ \ \textit{for every} \ \ z \in \mathbb{R}^J \ \ \textit{and every} \ \ \alpha \in A.$$

Proof. By contradiction. Let us assume that, for every $n \in \mathbb{N}$, there exist $z_n \in \mathbb{R}^J$, $\alpha_n \in A$ such that $\|W(\alpha_n)z_n\| < \frac{1}{n}\|z_n\|$. Noticing that $z_n \neq 0$, without any loss of generality we can assume that $\left(\frac{z_n}{\|z_n\|}\right)_n$ (which is in the unit sphere of $\mathbb{R}^J$) converges to some element $v \neq 0$ and (α_n) converges to some element $\alpha \in A$ (since A is compact). By the continuity of the map W, taking the limit when $n \to \infty$, we get $\|W(\alpha)v\| \leq 0$, hence $W(\alpha)v = 0$, a contradiction with the hypothesis that rank $W(\alpha) = \sharp J$. □

Proof of Proposition 3.1. Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ be fixed. We first show that, for every $i \in I$, the set $\hat{X}^i(\lambda)$ is bounded. Indeed, since the sets X^i are bounded below, there exist $\underline{x}^i \in \mathbb{R}^L$ such that $X^i \subset \underline{x}^i + \mathbb{R}^L_+$. If $x^i \in \hat{X}^i(\lambda)$, there exist $x^j \in X^j$, for every $j \neq i$, such that $\sum_{j\in J} x^j = \sum_{j\in J} e^j$. Consequently,

$$\underline{x}^i \leq x^i = -\sum_{j\neq i} x^j + \sum_{j\in J} e^j \leq -\sum_{j\neq i} \underline{x}^j + \sum_{j\in J} e^j$$

and so $\hat{X}^i(\lambda)$ is bounded.

We now show that $\hat{Z}^i(\lambda)$ is bounded under the three sufficient assumptions *(i)*, *(ii)* or *(iii)* of Proposition 3.1. Indeed, for every $z^i \in \hat{Z}^i(\lambda)$ there exist $(z^j)_{j\neq i} \in \prod_{j\neq i} Z^j$, $(x^j)_j \in \prod_{j\in I} X^j$, $p \in B_L(0,1)$, $q \in \mathbb{R}^J$ such that ${}^tW(p,q)\lambda \in B_J(0,1)$, $\sum_{j\in J} z^j = 0$ and $(x^j, z^j) \in B^j_{\mathcal{F}}(p,q)$.

Under Assumption (*i*), for every $j \in I$ the portfolio set Z^j is bounded from below, that is there exists $\underline{z}^j \in \mathbb{R}^J$ such that $Z^j \subset \underline{z}^j + \mathbb{R}^J_+$. Using the fact that $\sum_{j \in I} z^j = 0$, we get

$$\underline{z}^i \leq z^i = -\sum_{j \neq i} z^j \leq -\sum_{j \neq i} \underline{z}^j \ \text{ for every } \ z^i \in \hat{Z}^i(\lambda).$$

Under Assumption (*ii*), since $(x^i, z^i) \in B^i_{\mathcal{F}}(p, q)$ and $(x^i, p) \in \hat{X}^i(\lambda) \times B_L(0, 1)$, a compact set from above, there exists $\alpha^i \in \mathbb{R}^{\mathbb{D}}$ such that

$$\alpha^i \leq p \,\square\, (x^i - e^i) \leq W(p, q) z^i.$$

But (using the fact that $\sum_{i \in I} z^i = 0$) we also have

$$W(p, q) z^i = W(p, q)\Big(-\sum_{j \neq i} z^j\Big) \leq -\sum_{j \neq i} \alpha^j,$$

hence there exists $r > 0$ such that $W(p, q) z^i \subset B_{\mathbb{D}}(0, r)$.

From Lemma 5.1, taking $W(\alpha) = W(p, q)$ for $\alpha = (p, q) \in A := \big\{(p, q) \in B_L(0, 1) \times \mathbb{R}^J : {}^tW(p, q)\lambda \in B_J(0, 1)\big\}$, which is compact, for fixed $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, there exists $c > 0$ such that, for every $(p, q) \in A$, $z^i \in \mathbb{R}^J$, $c\|z^i\| \leq \|W(p, q) z^i\|$. Hence,

$$c\|z^i\| \leq \|W(p, q) z^i\| \leq r \ \text{ for every } \ z^i \in \hat{Z}^i(\lambda),$$

which shows that the set $\hat{Z}^i(\lambda)$ is bounded.

Finally, under Assumption *(iii)* the case of *short-lived assets* is a consequence of Part (*ii*) and Proposition 5.2.b. □

5.4 Proof of the no-arbitrage characterization Theorem 2.1

The proof is a direct consequence of the following result by taking $W := W_{\mathcal{F}}(p, q)$, $\bar{c} = z^i$ and $C = Z^i$.

Theorem 5.1 (Koopmans [19]). *Let $W : \mathbb{R}^n \to \mathbb{R}^m$ be linear, let $C \subset \mathbb{R}^n$ be convex, let $\bar{c} \in C$, and consider the two following assertions:*

(i) there exists $\lambda \in \mathbb{R}^m_{++}$ such that ${}^tW\lambda \in N_C(\bar{c})$,
or equivalently, $\lambda \bullet_m W\bar{c} = [{}^tW\lambda] \bullet_n \bar{c} \geq \lambda \bullet_m Wc = [{}^tW\lambda] \bullet_n c$ for every $c \in C$;

(ii) $W(C) \cap (W\bar{c} + \mathbb{R}^n_+) = \{0\}$.

The implication [(i) ⇒ (ii)] always holds and the converse is true under the additional assumption that C is a polyhedral set.

Proof of Theorem 5.1. $[(i) \Rightarrow (ii)]$ By contradiction. Suppose that there exists $c \in C$ such that $Wc > W\bar{c}$. This implies that, for every $\lambda \in \mathbb{R}^m_{++}$, $\lambda \bullet_m Wc > \lambda \bullet_m W\bar{c}$ or equivalently $[{}^tW\lambda] \bullet_n c > [{}^tW\lambda] \bullet_n \bar{c}$, that is, ${}^tW\lambda \notin N_C(\bar{c})$, which contradicts (i). □

For the proof of the implication $[(ii) \Rightarrow (i)]$, see Koopmans ([19]), taking into account the following known result on polyhedral sets.

Lemma 5.2. *Let $C \subset \mathbb{R}^n$ be a convex set.*
(a) ([33] Theorem 19.1) Then C is polyhedral if and only if there exist finitely many vectors $c_1, \ldots, c_k, d_1, \ldots, d_r$ in $\mathbb{R}^n$ such that

$$C = \text{co}\{c_1, \ldots, c_k\} + \left\{ \sum_{j=1}^{r} \beta_j d_j \;\middle|\; \beta_j \geq 0,\ j = 1, \ldots, r \right\}.$$

(b) ([33] Theorem 19.3) Let $W\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. If $C \subset \mathbb{R}^n$ is polyhedral set, then $W(C)$ is also polyhedral.

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