Financial economies with Restricted Participation

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Layout

Restricted participation

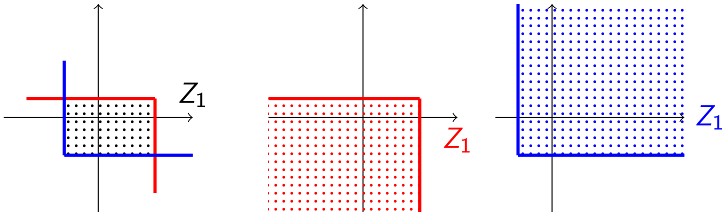
- Radner (1972), Siconolfi (1989), Cass (1984)

  > *"A very significant analysis from an interpretive viewpoint . . . is the imposition of institutional restrictions on trading activity in the bond (financial) markets. .. such restricted participation is to assume that in addition to the bugdet constraints, households face the financial constraints* $z_i \in Z_i \subset \mathbb{R}^J$ *for* $i \in I$*."*

- Elsinger and Summer (1999) for examples

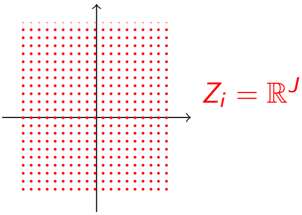
Example 1: Bounded Arbitrage AND $0 \in \mathrm{ri} Z_i$

Radner (1972)

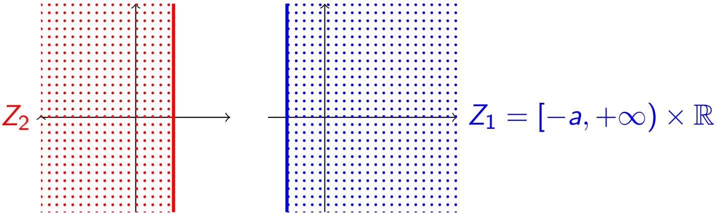


Example 2: Unbounded Arbitrage AND $0 \in \mathrm{ri} Z_i$

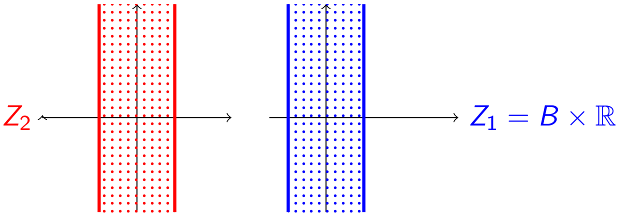
Unconstrained portfolios $Z_i = \mathbb{R}^J$ with nominal or numéraire assets
Cass-Duffie-Werner–Geanakoplos-Polemarchakis



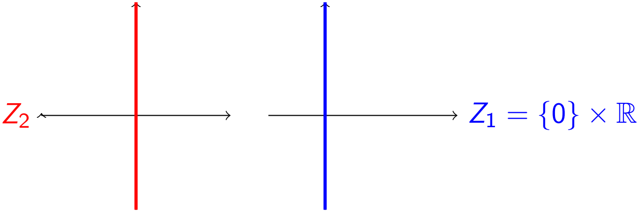
2 agents one and two, 2 assets



Example 2: Unbounded Arbitrage AND $0 \in \mathrm{ri} Z_i$



Restricted Participation is more of a norm than an exception
for example, $Z_i = \{0\} \times \cdots \times \{0\} \times \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}$
Hence is a cause why markets are incomplete



Example 2: Unbounded Arbitrage AND $0 \in \mathrm{ri} Z_i$

• a combination of the above cases (possibly different between agents)

$Z_i = \{0\}^{J_1} \times B_{J_2}(0, r) \times [\underline{z}_i + \mathbb{R}_+^{J_3}] \times \mathbb{R}^{J_4} \times$ vector space $\times \ldots$

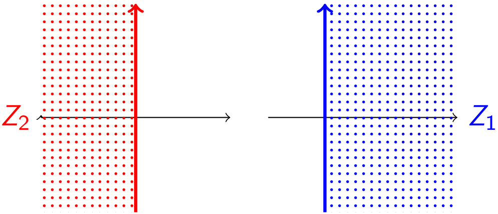
For all i, partition $J = J_1 \cup \cdots \cup J_k$ (depending on i)

• Linear inequalities [Aouani-Cornet] replaces boundedness: TODAY

Example 4: Unbounded Arbitrage AND $0 \notin \mathrm{ri} Z_i$

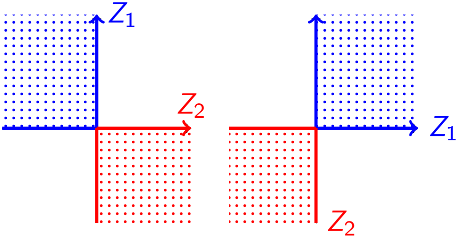
Generalized Cass Condition

2 agents one and two, 2 assets



Example 5: Unbounded Arbitrage AND $0 \notin \mathrm{ri} Z_i$

Sellers and Buyers



Example 6: Bid/Ask Spread

Example 1a

- $A^1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R},\ q \in \mathbb{R}$
- replaced by

$$V := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^2_+,\ z_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \geq 0,\ q = \begin{bmatrix} \overline{q} \\ -\underline{q} \end{bmatrix}$$

Example 6: Bid/Ask Spread

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- Notice the usual clearing condition $\sum_{i=1}^{I} z_i = 0$ has no more sense here [since it implies $z_i = 0$ for all i.]
- We will thus introduce Market Makers/Producers

Endogeneous restrictions

1. Endogeneous restrictions: $Z_i(q)$ depend on asset price q
   - Borrowing constraints

     $Z_i = \{z : q \cdot z \geq -\underline{\alpha}_i\}$ for given $\underline{\alpha}_i \geq 0$
   - Margin Requirements

     $Z_i = \{z \in \mathbb{R}^J \mid q^j z^j \geq -m^j q \cdot z\}$ for given $m_j \in \mathbb{R}_+$
   - Collateral Requirements

     $Z_i = \{z \in \mathbb{R}^J \mid q \cdot z^- \geq -\theta q \cdot z^+\}$ for given $\theta \in [0, 1]$

   Seghir-Torres-Martinez (2011), Villanacci et al (2011), Cornet (2011) Not today

Restricted participation

- Today restrictions on portfolios are exogenously given for institutional reasons but can be made endogeneous
- So Portfolio sets, Z_i, are taken as primitives of the economy assumed to be closed convex, $0 \in Z_i$ NOT bounded below

Restricted participation

- Today restrictions on portfolios are exogenously given for institutional reasons but can be made endogeneous
- So Portfolio sets, Z_i, are taken as primitives of the economy assumed to be closed convex, $0 \in Z_i$ NOT bounded below
- As consumption sets X_i are usually assumed to be closed convex, $\omega_i \in X_i$ AND bounded below
- bounded / unbounded arbitrage $\{(z_1, \ldots, z_I) : \forall i, z_i \in Z_i, \sum_{i=1}^{I} z_i = 0\}$ bounded / unbounded

References

1. Unconstrained case ($Z_i = \mathbb{R}^J$ is the whole space)
   Real Assets: Duffie-Shafer (1985), Geanakoplos-Mas Colell (1990), Geanakoplos-Shafer (1990), Hirsh-Magill-Mas Colell (1990), Husseini-Lasry-Magill (1990), Magill-Shafer (1990)
   Nominal assets : Cass (1984); Duffie (1985); Werner (1985)
   Numéraire assets : Geanakoplos-Polemarchakis (1986)

References

1. Unconstrained case ($Z_i = \mathbb{R}^J$ is the whole space)
2. Linear equality constraints (Z_i is a vector space)
   Balasko-Cass-Siconolfi (1990),
   Polemarchakis-Siconolfi (1997)

References

1. Unconstrained case ($Z_i = \mathbb{R}^J$ is the whole space)
2. Linear equality constraints (Z_i is a vector space)
3. Differentiable convex constraints
   Cass-Siconolfi-Villanacci (2001), Carosi-Gori-Villanacci (2009), Gori-Pireddu-Villanacci, (2011), Hoelle-Pireddu-Villanacci (2011) Seghir-Torres-Martinez (2011)

References

1. Unconstrained case ($Z_i = \mathbb{R}^J$ is the whole space)
2. Linear equality constraints (Z_i is a vector space)
3. Differentiable convex constraints
4. Z_i defined by closed convex sets containing zero
   Radner (1972), Cass (1984), Siconolfi (1989), Angeloni-Cornet (2006), Da-Rocha-Triki (2004, 2005), Hahn-Won (2007), Cornet-Gopalan (2008), Aouani-Cornet (2009, 2010), Cornet-Ranjan (2011a,b)

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2. Linear equality constraints (Z_i is a vector space)
3. Differentiable convex constraints
4. Z_i defined by closed convex sets containing zero
5. Unbounded arbitrage literature
   Hart (1974), Werner (1987)
   Chichilnisky 1993, 1994a, l994b, 1995a, 1995b
   Alouch-Le Van-Page 2002, Dana-Le Van-Magnien 1999, le Van-Page-Wooders 2003, Milne 1976, 1980, Nielsen 1989, Page 1959,1982, 1987, 1996, Page-Schlesinger 1993, Page-Wooders 1993,1996, 2000, Page-Wooders-Monteiro 2000.

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2. Linear equality constraints (Z_i is a vector space)
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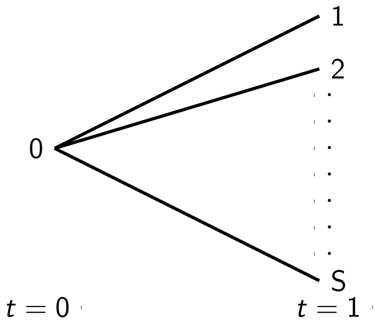
4. Z_i defined by closed convex sets containing zero
5. Unbounded arbitrage literature
6. Bid-Ask Spread

Layout

Time and Uncertainty with 2 dates



- Finite set of nodes: $\bar{S} = \{0, 1, 2, \ldots, S\}$.
- Spot markets for goods at each state: $\{1, 2, \ldots, \ell\}$
- Commodity space is $\mathbb{R}^L$ with $L = \ell(S + 1)$
- Consumptions $x \in \mathbb{R}^L$ and commodity prices $p \in \mathbb{R}^L$

Financial Assets

- Set of assets: $\{1, 2, \ldots, J\}$
- Payoff matrix : $S \times J$ matrix $V(p)$ of columns $V^j(p)$
- Portfolios z and asset prices q belong to $\mathbb{R}^J$
- Each consumer $i \in \{1, 2, ..., I\}$ is characterized by $(X_i, \omega_i, u_i, Z_i)$.

Budget sets

Budget set

$B_i(p)$ is the set of $(x, z) \in X_i \times Z_i$ such that

$$\begin{aligned} p(0) \cdot x(0) &\leq p(0) \cdot e_i(0) \qquad \text{for } s = 0 \\ p(s) \cdot x(s) &\leq p(s) \cdot e_i(s) \qquad \text{for } s \in S \end{aligned}$$

Budget sets

Budget set

$B_i(p, q)$ is the set of $(x, z) \in X_i \times Z_i$ such that

$$\begin{aligned} p(0) \cdot x(0) + q \cdot z &\leq p(0) \cdot e_i(0) \qquad \text{for } s = 0 \\ p(s) \cdot x(s) &\leq \sum_{\jmath=1}^{J} V_s^j(p) z^j + p(s) \cdot e_i(s) \qquad \text{for } s \in S \end{aligned}$$

Assumptions on Consumption and Financial sectors

Assumption C

For all consumer $i \in I$

- $X_i = \mathbb{R}^L_+$
- u_i is strongly monotonic, quasi-concave, continuous
- $e_i \geq 0$

Assumption F on $\mathcal{F} = (V, (Z_i)_i)$

- $Z_i \subset \mathbb{R}^J$ is closed, convex, and $0 \in Z_i$ for every i
- **Either** $\mathcal{F}$ is nominal, i.e., $V(p) = R$ is independent of p
- **Or** $\mathcal{F}$ consists of numéraire assets with good ℓ for numéraire

Financial Equilibrium

$(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium if

1. Commodity and asset markets clear:
   $\sum_{i=1}^{I} \bar{x}_i = \sum_{i=1}^{I} e_i$ and $\sum_{i=1}^{I} \bar{z}_i = 0$
2. For all $i \in I, (\bar{x}_i, \bar{z}_i)$ maximizes u_i in $B_i(\bar{p}, \bar{q})$

Layout

Nominal Assets

- Cass, D., (1984) (2006)
- Da Rocha and Triki (2005)
- Florig and Meddeb (2007)
- Cornet and Gopalan (2010)

Arbitrage-free prices

1. Aggregate Arbitrage-Free $q \in Q_{\mathrm{ag}}$ if
   there is no $z \in \mathbb{R}^J, \; W(\bar{q})z > 0$
2. (Individually) Arbitrage-Free $q \in Q$ if
   for every agent i, there is no $z_i \in \mathbf{A}Z_i, \; W(\bar{q})z_i > 0$

- $Q_{\mathrm{ag}} \subset Q$,

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Lemma

$Q_{\mathrm{ag}} = Q$ *under the assumption*

Generalized Cass Survival

$\bigcup_{i \in \mathbb{I}} Z_i = \mathbb{R}^J$ *(each portfolio is accessible by some agent)*

Moreover (equilibrium asset prices) $E_{\mathrm{q}} \subset Q_{\mathrm{ag}} = Q$

holds if $\exists i \in I, \; Z_i = \mathbb{R}^J$ unconstrained portfolios

Layout

Main Result for Nominal Assets

Theorem

Under **C**, **F** only nominal assets and **Generalized Cass Survival**
$Q_{\mathrm{ag}} = E_q = Q$

- $Q_{\mathrm{ag}} \subset E_q$ [Main Result]

Generalized Cass Survival

1. Cass Survival: for all i, $Z_i \subset Z_1$ and $Z_1 = \mathbb{R}^J$
2. Generalized Cass Survival: $\bigcup_{i \in I} Z_i = \mathbb{R}^J$

Generalized Cass Survival

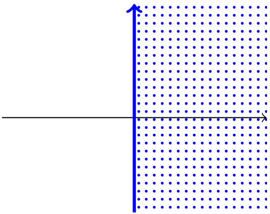
1. Cass Survival: for all i, $Z_i \subset Z_1$ and Z_1 vector space
2. Generalized Cass Survival: $\bigcup_{i \in I} Z_i$ vector space

Examples satisfying or not Generalized Cass Survival

Two assets and two (groups of) agents.

Agent 1 can only buy the asset 1. Agent 2 can only sell asset 1

Each agent can either sell or buy Asset 2



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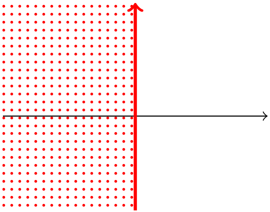

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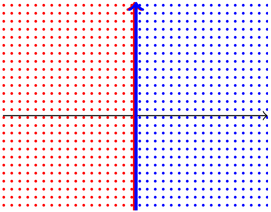

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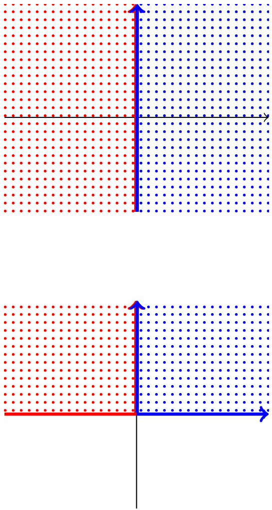

Layout

Layout

Existence of equilibria with Bounded Arbitrage

Theorem: Radner (1972)

$(\mathcal{E}, \mathcal{F})$ has an equilibrium under **C**, **F**, and
$e_i \in \text{int}X_i$ and $Z_i = B(0, r_i)$ for all i CAN BE CUT IN TWO

Existence of equilibria with Bounded Arbitrage

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$e_i \in \mathrm{int} X_i$ and $Z_i = B(0, r_i)$ for all i CAN BE CUT IN TWO

- **B** Bounded Arbitrage, i.e., bounded admissible portfolios

$$\mathcal{A}(v) := \{(z_1, \ldots, z_I) : \forall i, z_i \in Z_i, Rz_i \geq v_i, \sum_i z_i = 0\}$$

- **S** Survival: $e_i \in \mathrm{int} X_i$ and $0 \in \mathrm{int}\ Z_i$
- Goal is to
  1. weaken **S** to cover the introductory examples
  2. remove assumption **B** and only assume linear constraints

Layout

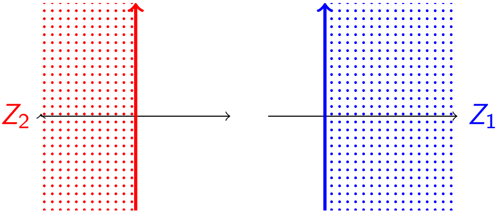
$0 \in \text{ri } Z_i$ better than $0 \in \text{int } Z_i$

- **S** Survival: $e_i \in \text{int}X_i$ and $0 \in \text{ri } Z_i$ $[\Leftarrow 0 \in \text{int } Z_i]$
- Not symmetric, i.e., does not work with $e_i \in \text{ri } X_i$ (example by Gale).
- $Z_i = \{0\}^{J_1} \times B_{J_2}(0, r) \times [\underline{z}_i + \mathbb{R}_+^{J_3}] \times \mathbb{R}^{J_4} \times$ vector space

Layout

Beyond $0 \in \mathrm{ri} Z_i$

$0 \in$ ri Z_i may not be satisfied under Cass Generalized Survival



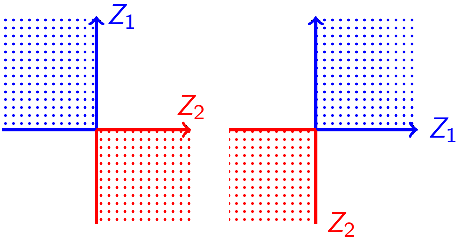
Weak Survival Assumption

S_w $\quad e_i \in$ int X_i for all i $\quad 0 \in \mathrm{ri}[\sum_{i=1}^{I} Z_i]$ [$\Leftarrow$ Cass]

Similar to the consumption side survival assumption
$\sum_{i=1}^{I} e_i \in$ int $[\sum_{i=1}^{I} X_i]$ $\quad [\sum_{i=1}^{I} e_i \gg 0$ when $X_i = \mathbb{R}_+^L]$

Beyond $0 \in \text{ri } Z_i$: "Sellers and Buyers"

- Banks: lenders and borrowers
- Insurance companies: insurers and insurees
- Betting markets: betters and bookmakers
- Separate the buying and selling accounts: Bid / Ask spread



Existence of quasi-equilibria

Corollary

$(\mathcal{E}, \mathcal{F})$ has an equilibrium under $\mathbf{B}$ and weak survival $\mathbf{S}_w$

Existence of quasi-equilibria

Corollary

$(\mathcal{E}, \mathcal{F})$ has an equilibrium under **B** and weak survival $\mathbf{S}_w$

Theorem

$(\mathcal{E}, \mathcal{F})$ has a quasi-equilibrium under under **B** [without $\mathbf{S}_w$].

Quasi-equilibrium is borrowed from

- Gottardi-Hens (1995)
- Seghir-Triki-Kanellopoulou (2001)
- Cornet-Ranjan (2011) presented at this conference

Layout

Unbounded Arbitrage

Goal is to remove Boundedness Assumption **B** and only assume Z_i defined by linear constraints

Theorem

$(\mathcal{E}, \mathcal{F})$ has an equilibrium under Weak Survival $\mathbf{S}_w$

Finance economy

- Finance economy $\mathcal{F} := ((X_i, u_i, e_i)_i, (Z_i)_i, R)$ with
- 2 dates $t = 0$ and $t = 1$, with S states at $t = 1$
- 1 good at each state (today and tomorrow)
- J numéraire assets

Finance economy

- Finance economy $\mathcal{F} := ((X_i, u_i, e_i)_i, (Z_i)_i, R)$ with
- 2 dates $t = 0$ and $t = 1$, with S states at $t = 1$
- 1 good at each state (today and tomorrow)
- J numéraire assets

Following Hart,

- Transform the financial economy in an exchange economy
- One-to-one correspondence between financial equilibria and Walras equilibria

Hart's trick

$$
\begin{aligned}
(\bar{x}_i, \bar{z}_i) \in \quad & \text{Argmax } u_i(x_i) \text{ subject to} \\
& \bar{p}(0)x_i(0) + \bar{q} \cdot z_i \leq \bar{p}(0)e_i(0) \\
& p(s)x_i(s) \leq (=)p(s)e_i(s) + p(s)R_s \cdot z_i, \ \ \forall s = 1, \ldots, S \\
& x_i \geq 0, \quad \text{and} \quad z_i \in Z_i.
\end{aligned}
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\end{aligned}
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\begin{aligned}
(\bar{x}_i(0), \bar{z}_i) \in \quad & \text{Argmax} && u_i(x_i(0), e_i(1) + R_1 \cdot z_i, \dots, e_i(S) + R_S \cdot z_i) \\
& && := U_i(x_i(0), z_i), \ \text{subject to} \\
& \bar{p}(0)x_i(0) && + \bar{q} \cdot z_i \leq (\bar{p}(0), \bar{q}) \cdot (e_i(0), 0) \\
& (x_i(0), z_i) && \in C_i \\
& x_i(0) \geq 0, && z_i \in Z_i, e_i(s) + R_s \cdot z_i \geq 0, \quad \text{for } s = 1, \dots, S
\end{aligned}
$$

- One-to-one correspondence between
  - Financial equilibria of $\mathcal{F}$
  - Walras equilibria of $\mathcal{E}_{\mathcal{F}} = (C_i, U_i, \omega_i), \omega_i := (e_i(0), 0)$

Layout

Existence of financial equilibria

1. Admissible utility set $\mathcal{U}$ of $\mathcal{E}_{\mathcal{F}}$ is compact

   $\mathcal{U} = \{(U_1(x_1), \ldots, U_I(x_I)) : \forall i, x_i \in C_i \sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i\}$

2. Get existence of an (quasi-)equilibrium of $\mathcal{E}_{exch}$ (hence of $\mathcal{E}_{\mathcal{F}}$) when $\mathcal{U}$ is compact from Dana-Le Van-Magnien (1999)
3. Hence existence of financial equilibria of $\mathcal{F}$
4. Hart's trick works with portfolio constraints Z_i
5. But only works with 1 good and 2 dates

Layout

Why eliminate "redundant" assets?

In the payoff space $\mathbb{R}^S$ the following set is bounded

$$\{(Vz_1, \ldots, Vz_I) : \forall i, z_i \in Z_i, Rz_i \geq v_i, \sum_i z_i = 0\}$$

(and closed when the Z_i are polyhedral convex sets). Hence with no redundant assets (i.e., V one-to-one) in the portfolio space

B Bounded Arbitrage, i.e.,
In the portfolio space $\mathbb{R}^J$ the following set is bounded

$$\mathcal{A}(v) := \{(z_1, \ldots, z_I) : \forall i, z_i \in Z_i, Rz_i \geq v_i, \sum_i z_i = 0\}$$

Thus existence of equilibria follows from Radner.

Standard elimination of "redundant" assets

Standard arguement in the unconstrained case

- first eliminate redundant assets

Suppose $I = 3, S = 2, J = 3$, and define $\mathcal{F}$ by

$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $Z_i = \mathbb{R}^3$ for all i

We define $\mathcal{F}'$ by removing the bond which is redundant, that is,

$V' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Z_i = \mathbb{R}^2$ for all i

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- Check that $\mathcal{F}' \sim \mathcal{F}$, that is, $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ have the same consumption equilibria for every standard economy $\mathcal{E}$.

- Then $(\mathcal{E}, \mathcal{F}')$ has equilibria from Radner since $\mathcal{F}'$ bounded arbitrage hence $(\mathcal{E}, \mathcal{F})$ has equilibria since $\mathcal{F}' \sim \mathcal{F}$.

Does not work with constraints

Suppose $I = 3, S = 2, J = 3$, and $V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\begin{aligned} Z_1 &= \mathbb{R}^2 \times \{0\}, [\text{or } Z_1 = \mathbb{R}^3] \\ Z_2 &= \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \beta = 0, \alpha = 2\gamma\}, \\ Z_3 &= \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha = 0, \beta = 3\gamma\} \end{aligned}$$

Does not work with constraints

Suppose $I = 3, S = 2, J = 3$, and $V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\begin{aligned} Z_1 &= \mathbb{R}^2 \times \{0\}, [\text{or } Z_1 = \mathbb{R}^3] \\ Z_2 &= \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \beta = 0, \alpha = 2\gamma\}, \\ Z_3 &= \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha = 0, \beta = 3\gamma\} \end{aligned}$$

• Removing the bond (which is called redundant when there is no constraint), means that $\gamma = 0$
$\Rightarrow Z_1' = \mathbb{R}^3, Z_2' = \{0\}, Z_3' = \{0\}$ hence "kills" the asset market.
because at equilibrium $\bar{z}_1' + \bar{z}_2' + \bar{z}_3' = 0 \Rightarrow \bar{z}_1' = \bar{z}_2' = \bar{z}_3' = 0$

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• Similarly, removing the first (or the second one) kills also the asset market

Redundant assets

- Balasko, Cass, and Siconolfi (1990)

  > *"One significant source of restricted participation is financial intermediation (...), which typically involves redundancy."*

- Cass, Siconolfi, and Villanacci (2001)

  > *"In this context, (Nonredundant) Assumption 1 is not at all innocuous. When their portfolio holdings are constrained, households may very well benefit from the opportunities afforded by the availability of additional bonds whose yields are not linearly independent."*

Eliminating the redundant bond

$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $Z_i = \mathbb{R}^3$ for all i

is equivalent to replacing $Z_i = \mathbb{R}^3$ by $Z_i' = \mathbb{R}^2 \times \{0\}$

hence removing useless portfolios (in a precise sense)

keeping the same payoff matrix

Eliminating Werner useless portfolios

$$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} Z_1 &= \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha + \gamma = 0, \beta + \gamma = 0\}, \\ Z_2 &= \mathbb{R}^3, Z_3 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha = \beta = 0\}. \end{aligned}$$

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Above $Z_1 \cap$ ker $V =$ ker V (since $Z_1 =$ ker V) and are useless.

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(With linear spaces) z_i is (Werner) useless if $z_i \in Z_i \cap \ker V$ ($\Rightarrow q \cdot z_i = 0$ at equilibrium).

Above $Z_1 \cap \ker V = \ker V$ (since $Z_1 = \ker V$) and are useless. We eliminate useless portfolios to get $\mathcal{F}'$ "reduced" such that

$\forall i, \quad Z_i' \cap \ker V = \{0\}.$

$\mathcal{L}_{\mathcal{F}'} := \sum_{i=1}^{I} (Z_i' \cap \ker V) = \{0\}.$

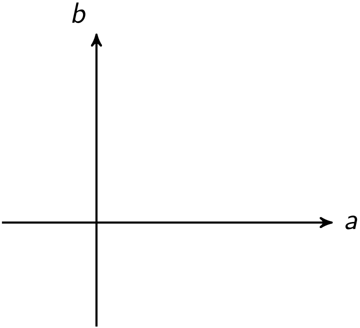
Layout

References

- Hahn-Wan (2007)
- Aouani-Cornet (2009, 2010, 2011)

Useles portfolios

Consider two agents, two states and two assets.



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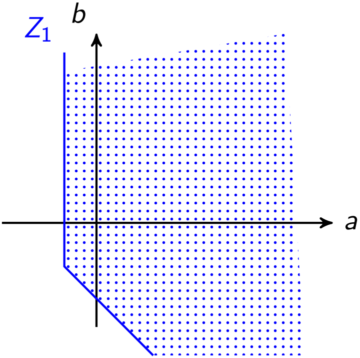

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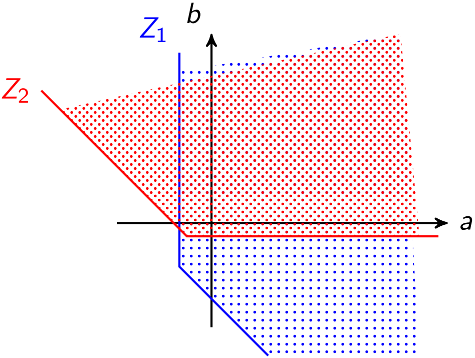
Consider two agents, two states and two assets.

No Werner useless portfolios: no vector space
$L \subset Z_i \cap \ker V \subset \ker V$



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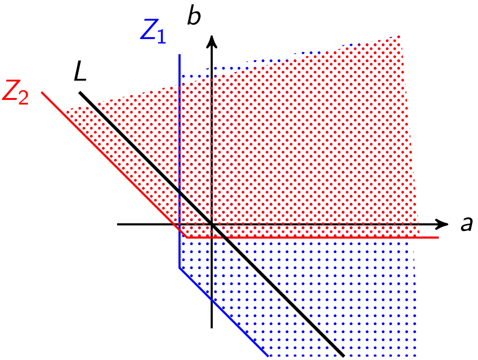

Let a vector space $L \subset \sum_{i \in I} Z_i \cap \ker V \subset \ker V$ $\quad (*)$

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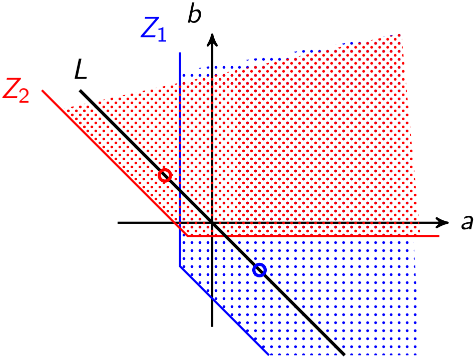
Consider two agents, two states and two assets.
UNBOUNDED ARBITRAGE



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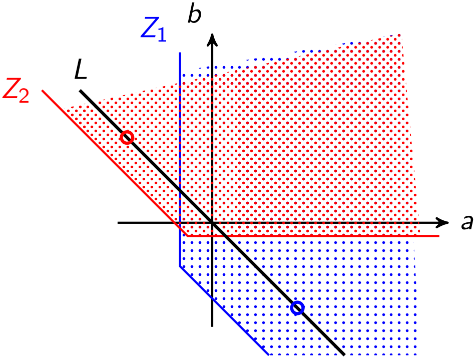

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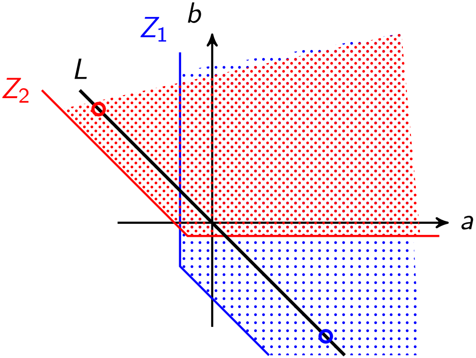

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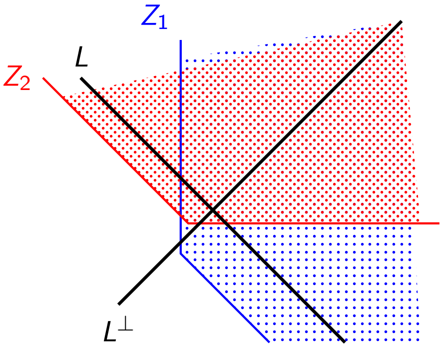

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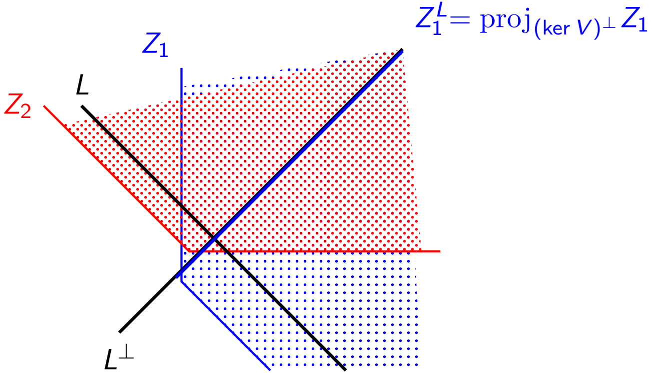

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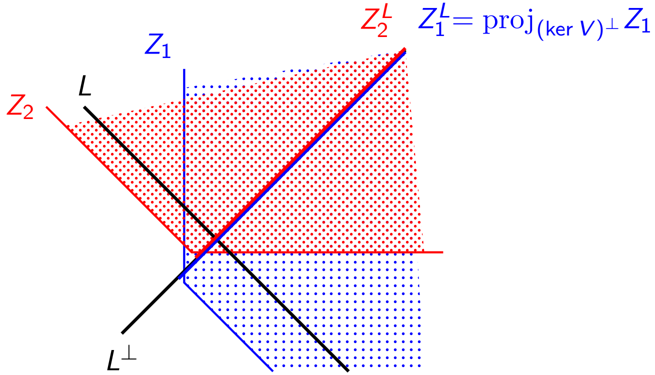

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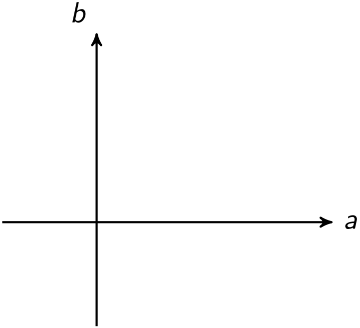

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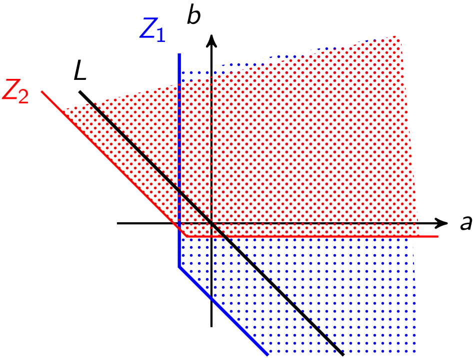

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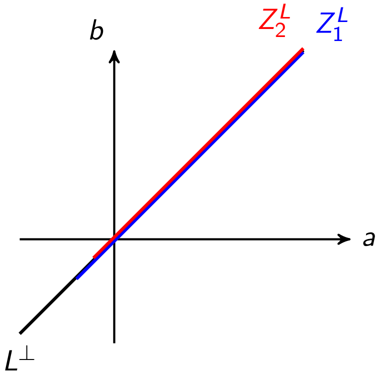

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Useles portfolios

Theorem 1

$\mathcal{F}^L \sim \mathcal{F}$ for every L as in $(*)$
i.e., $\forall \mathcal{E}$ $(\mathcal{E}, \mathcal{F}^L)$ and $(\mathcal{E}, \mathcal{F})$ same consumption equilibria.

Useles portfolios

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If $\mathbf{L}$ is the greatest vector space satisfying $(*)$
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So we can find a bounded $\mathcal{F}'$ which is equivalent to $\mathcal{F}$

- either in one step with $\mathcal{F}' = \mathcal{F}^{L}$, L is of maximal dimension
- or sequentially $L^1 \subset \mathbf{L}(\mathcal{F})$, $L^2 \subset \mathbf{L}(\mathcal{F}^{L^1}), \ldots$

Layout

1a. Separating the buying and selling accounts

- $A^1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R},\ q \in \mathbb{R}$
- replaced by

$$V := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}_+ \times \mathbb{R}_+,\ z_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \geq 0,\ q = \begin{bmatrix} \overline{q} \\ -\underline{q} \end{bmatrix}$$

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- $q \cdot y = (\overline{q}, -\underline{q}) \cdot (a, b) = (\overline{q} - \underline{q})a$
- Arbitrage-free (under NS) at equilibrium $\Rightarrow \overline{q} \geq \underline{q}$
- $\overline{q} = \underline{q}$ iff $\bar{y}$ maximizes profit $\bar{q} \cdot y$ subject to $y \in Y$

No perfect matching between sellers and buyers

Example 2a

$$V := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^3_+, \; z_i = \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix}, \; q = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix}$$

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We notice that the usual portfolio clearing condition $\sum_{i=1}^{I} z_i = 0$ has no more sense in this model and is replaced by

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Arbitrage-free (under non-satiation) at equilibrium implies (since
$V1 = 0 \Rightarrow$
$q^1 + q^2 + q^3 \geq 0 \iff \overline{q} := q^1 + q^2 \geq \underline{q} := -q^3$
[???$q \cdot y \geq 0$ for all $y \in Y := \ker V \cap \mathbb{R}^2_+$]

No perfect matching between sellers and buyers

Example 2b

$$V := \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Same type of example as before but with the possibilty of 2 "independent" Market Maker and $\dim \ker V = 2$

Who is buyer and who is seller?

Example 3

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Example 4

$V := \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^6_+, \; z_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \; q = \begin{bmatrix} \overline{q} \\ -\underline{q} \end{bmatrix}$

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$V(\alpha, \beta) = 0 \iff \alpha^1 + \alpha^3 = \beta^1 + \beta^3$ and $\alpha^2 + \alpha^3 = \beta^2 + \beta^3$

Note that $V(1,1,1,2,2,0) = 0$ with $\alpha \neq \beta$

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- $\sum_{i=1}^I \alpha_i = \sum_{i=1}^I \beta_i$ and $\sum_{i=1}^I \alpha_i + \sum_{i=1}^I \gamma_i = \sum_{i=1}^I \delta_i$ $\iff V(\sum_{i=1}^I z_i) = 0$

$\iff \sum_{i=1}^I z_i \in \{(a, b, c, d) \in \mathbb{R}^4_+ : a = b, a + c = d\} = \ker V \cap \mathbb{R}^4_+ := Y$

$q \cdot (\sum_{i=1}^I z_i) = (\overline{q}, -\underline{q}) \cdot (\sum_{i=1}^I \alpha_i, \sum_{i=1}^I \beta_i) = (\overline{q} - \underline{q}) \sum_{i=1}^I \alpha_i$

Arbitrage-free (under non-satiation) at equilibrium implies $\overline{q} \geq \underline{q} \iff q^1 + q^2 \geq 0$ [$q \cdot y \geq 0$ for all $y \in Y := \ker V \cap \mathbb{R}^2_+$]

Example 4

$$V := \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^3_+, \; z_i = (\alpha_i, \beta_i, \gamma_i, \delta_i), \; q = (r, s, t, u)$$

Example 4

$$V := \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^3_+, \ z_i = (\alpha_i, \beta_i, \gamma_i, \delta_i), \ q =$$

(r, s, t, u)

Example 4

$$V := \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^3_+, \quad z_i = (\alpha_i, \beta_i, \gamma_i, \delta_i), \; q = (r, s, t, u)$$

Layout

Market makers/production sets

Financial structure with Market Makers

$$\mathcal{F} = \Big(S, J, V, (Z_i)_{i\in I}, (Y_k)_{k\in K}, (\theta_{ik})_{i\in I, k\in K}\Big)$$

Several Market makers $k = 1, \ldots, K$ each of which

• is represented by its "production set" $Y_k \subset \mathbb{R}^J$

• is profit maximizing, i.e.,
chooses $y_k^* \in Y_k$ so that $q^* \cdot y_k^* = \max\{q^* \cdot y_k : y_k \in Y_k\}$

• and profits of the Market Makers are redistributed to the consumers according to their shares θ_{ik}

If $Y = \{0\}$, i.e., No Market Maker, then $\mathcal{F}$ simply denoted

$$\mathcal{F} = \Big(S, J, V, (Z_i)_{i\in I}\Big)$$

Equilibrium

Definition

$(x^*, p^*, y^*, z^*, q^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$ if

1. For all $i \in I, (x_i^*, z_i^*)$ maximizes u_i in $B_i(p^*, q^*, \pi_i^*)$ with $\pi_i^* := \sum_{k=1}^{K} \theta_{ik} q^* \cdot y_k^*$ [$= 0$ if Y_k is a cone $\forall k$]
2. $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i$ [Commodity markets clear]
3. $\sum_{i=1}^{I} z_i^* = \sum_{k=1}^{K} y_k^*$ [Asset portfolios clear]
4. For all $k \in K$, [Market Makers Profit Maximization]
   $y_k^* \in Y_k$ and $q^* \cdot y_k^* = \max\{q^* \cdot y_k : y_k \in Y_k\}$
   $y^* \in Y$, $\pi^* := q^* \cdot y^* = 0$ [No profit] and $q^* \cdot y \leq 0, \ \forall y \in Y$

Existence of equilibria 1

Theorem

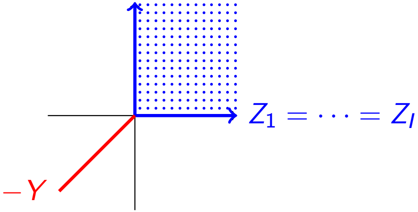
There exists an equilibrium under assumptions **C**, **F** *and*

MM $\forall k \; Y_k$ *is a closed convex cone* *OR* $Y = \sum_{k \in K} Y_k$ *is a closed convex cone*

S $0 \in \mathrm{ri}[\sum_{i=1}^{I} Z_i - Y]$ *and* $e_i \gg 0$ *for all* i.

Survival Assumption **S**

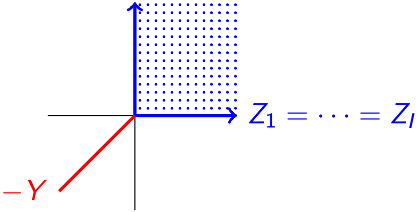
$0 \in \mathrm{ri}[\sum_{i=1}^{I} Z_i - Y]$



$$V := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^2_+, \; z_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \geq 0, \; q = \begin{bmatrix} \overline{q} \\ -\underline{q} \end{bmatrix}$$

Survival Assumption **S**

$0 \in \mathrm{ri}[\sum_{i=1}^{I} Z_i - Y]$



$$V := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \forall i, Z_i := \mathbb{R}^2_+, \; z_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \geq 0, \; q = \begin{bmatrix} \overline{q} \\ -\underline{q} \end{bmatrix}$$

$Y := \{(a,b) \in \mathbb{R}^2_+ : a = b\} := \ker V \cap \mathbb{R}^2_+$

What about $Y := \{(a,b) \in \mathbb{R}^2 : a = b\} := \ker V \cap \mathbb{R}^2_+$

Budget set

Budget set

$B_i(p^*, q^*)$ is the set of $(x_i, z_i) \in X_i \times Z_i$ such that

$$p^*(0) \cdot x_i(0) + q^* \cdot z_i \leq p^*(0) \cdot e_i(0)$$

$$p^*(s) \cdot x_i(s) \leq p^*(s) \cdot e_i(s) + V(s) \cdot z_i \text{ for all } s$$

Budget set

Budget set

$B_i(p^*, q^*, \pi_i)$ is the set of $(x_i, z_i) \in X_i \times Z_i$ such that

$p^*(0) \cdot x_i(0) + q^* \cdot z_i \leq p^*(0) \cdot e_i(0) + \pi_i$

$p^*(s) \cdot x_i(s) \leq p^*(s) \cdot e_i(s) + V(s) \cdot z_i$ for all s

Single Market Maker with cone production set

An important case to be considered hereafter is the case of

- single market maker (i.e., $K = 1$)
- production set (Y_1 simply denoted) Y is a closed convex cone

Thus Profit Maximization is equivalent to

$y^* \in Y$, $q^* \cdot y^* = 0$ and $q^* \cdot y \leq 0$ for all $y \in Y$

Note that the third blue condition is equivalent to

$q^* \in Y^o := \{q \in \mathbb{R}^J : q \cdot y \leq 0, \ \forall y \in Y\}$

Equilibrium with a single Market maker

If Y is a closed convex cone

$(x^*, p^*, y^*, z^*, q^*)$ is an equilibrium iff

1. For all $i \in I, (x_i^*, z_i^*)$ maximizes u_i in $B_i(p^*, q^*, 0 = \theta_i \pi^*)$
2. $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i$ [Commodity markets clear]
3. $\sum_{i=1}^{I} z_i^* = y^*$ [Asset markets clear]
4. $y^* \in Y$, $\pi^* := q^* \cdot y^* = 0$ [No profit] and $q^* \cdot y \leq 0, \ \forall y \in Y$

Layout

Theorem

$\mathcal{F} \sim (\mathcal{F}, Y)$ if

Y closed convex cone $Y \subset -\sum_{i=1}^{I} \mathbf{A} Z_i \cap \{V \geq 0\} + (Z_{\mathcal{F}})^{\perp}$,

More precisely let $\mathcal{E}$ be a standard exchange economy.

(a) Let (x^*, p^*, z^*, q^*) be an equilibrium of $(\mathcal{E}, \mathcal{F})$ then $(x^*, p^*, 0, z^*, q^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F}, Y)$.

(b) Conversely, let $(x^*, p^*, y^*, z^*, q^*)$ be an equilibrium of $(\mathcal{E}, \mathcal{F}, Y)$, then there exist $\bar{\bar{z}}_i \in Z_i$ $(i \in I)$ such that $(x^*, p^*, \bar{\bar{z}}, q^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

Layout

Accounts-clearing equilibria

Let $\mathcal{F} = (S, J, V, (Z_i)_{i \in I})$

(x^*, p^*, z^*, q^*) is an account-clearing equilibrium of $(\mathcal{E}, \mathcal{F})$ if

1. For all $i \in I, (x_i^*, z_i^*)$ maximizes u_i in $B_i(p^*, q^*)$
2. $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i$ [Commodity markets clear]
3. $W(q^*)(\sum_{i=1}^{I} z_i^*) = 0$ [Payoffs/Accounts clear]