Single-dipped Preferences with Indifferences: Strong Group Strategy-proof and Unanimous Social Choice Functions

Grisel Ayllón Aragón and Diego M. Caramuta*

Universitat Autònoma de Barcelona

July 29, 2011

Abstract

We study the problem of locating a public bad which generates severe negative externalities such as a dumpsite. We characterize the family of strong group strategy-proof and unanimous social choice functions when agents have single-dipped preferences with indifferences. The dip of the agent is for him the worst alternative for the location of the public bad. There might exist a location where each agent does not perceive the negative externalities of the public bad and become indifferent from that location onward. The range of a strong group strategy-proof and unanimous social choice function under this domain is richer than under single-dipped preferences, since locations not necessarily in the extremes of the alternative space may be chosen.

1 Introduction

The location of a public bad is a problem which concerns the agents that will be affected by its existence. Each of them can have different preferences over where to locate the

*We thank the financial support from "Ministerio de Ciencia e Innovación", project *ECO2008-04756*, "Grupo Consolidado" type C and FEDER.

public bad, making this social decision a difficult task. Where to place the public bad when the opinions about the the best location differ? There have been several attempts to solve this problem by proposing voting procedures under different preferences assumptions. We study the problem of locating a public bad under *single-dipped preferences with indifferences* asking two appealing properties to the social choice function: *strong group strategy-proofness and unanimity.* Strong group strategy-proofness is a property which will avoid the misrepresentation of the preferences. It asks the social choice function to rule out situations where a group of agents gain or at least some of them stay indifferent while the others gain with the outcome generated by changing their true preferences. Unanimity requires that if there exists a location that is preferred by all the agents then this location must be chosen.

We consider situations where the public bad causes negative externalities such as pollution, noise, radioactivity, bad smells, or even illnesses. For instance, we can think about a dumpsite which is necessary to be installed but causes severe negative externalities. Assume that n agents live along a segment of the real line $[0, T] \subset \mathbb{R}$. If an agent is asked to state his preferences about the location of the dumpsite, then the worst alternative might be that it could be installed where he lives and the furthest from home the public bad is located, the best for the agent. This reasoning leads us to define the *single-dipped* preferences where the location which gives the least level of satisfaction is called the "dip" and the agent strictly prefers any location further from it.

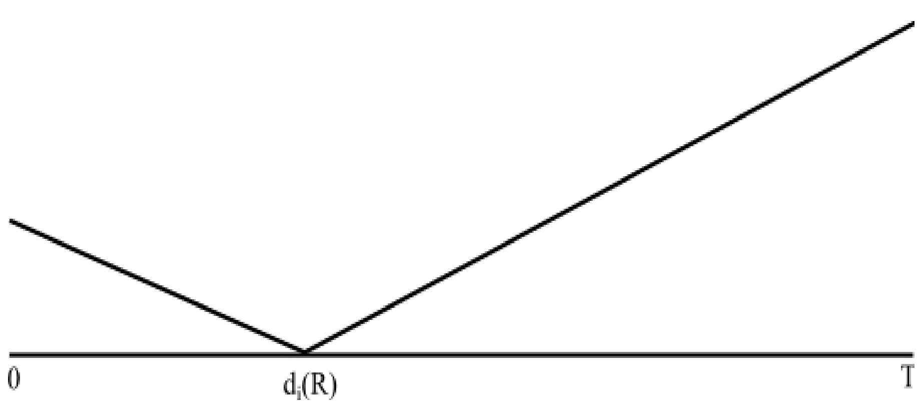

Figure 1. Single-dipped preferences.

Nevertheless, single-dipped preferences exclude very interesting situations. Consider the case where each agent may have a location to the left of his dip and/or to the right of his dip, such that the following happens: as we move away in one direction and/or the other from the dip of the agent, his welfare increases up to this location and remains constant from that position onwards. For example, there is an agent who lives near a mountain such as in the following figure. The utility of placing the dumpsite to the right hand side of his dip will increase up to a point where the mountain prevents the agent to perceive the pollution, from that position onwards the agent is indifferent about where to locate the public bad. We called this type of preferences "*single-dipped preferences with indifferences*".

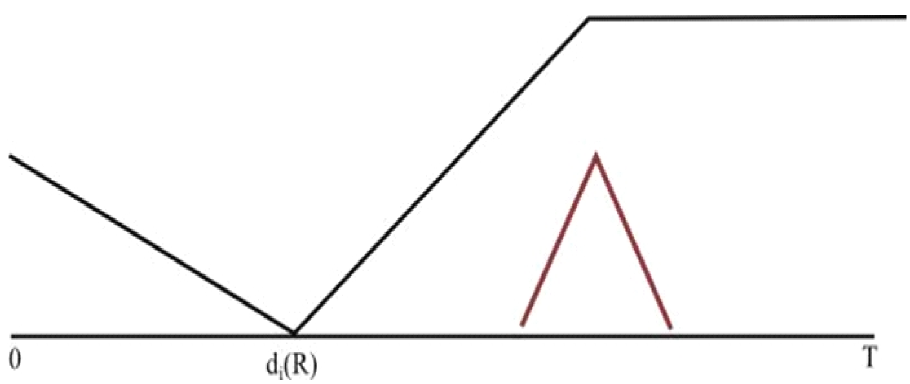

Figure 2. An agent is indifferent in the location of a dumpsite when he does not perceive the negative externalities.

Under this scenario a natural question arises: can we characterize the family of strong group strategy-proof and unanimous social choice functions under single-dipped preferences with indifferences? We answer this question in a positive way providing a class of social choice functions called "*full agreement rules*".

Manjunath (2009) studied the location of a public bad under the assumption of single-dipped preferences along a closed interval. He characterized the class of efficient and strategy-proof social choice functions. This result shows that the range of an efficient and strategy-proof social choice function must be the two extremes of the closed interval. Barberá, Berga, and Moreno (2010) proved that the range of strategy-proof social choice functions with single-dipped preferences contains two alternatives at most. Returning to our example of the dumpsite, if agents have single-dipped preferences along the line $[0, T]$, then the chosen location must be on 0 or T if an efficient and strategy-proof social choice function is applied. Now imagine that there are only two agents with single-dipped preferences with indifferences, whose dips are located near 0 and/or T. Under this new situation, why should the social choice function locate the dumpsite in any of the extremes and not in the middle where the public bad may be far enough from both agents? Given our extension of the preferences, we show that the range of the strong group strategy-proof and unanimous social choice functions is larger than the one obtained under the context of single dipped-preferences.

On the other hand, Manjunath (2009) showed that under single-dipped preferences strong group strategy-proofness is equivalent to strategy-proofness. Barberá, Berga, and Moreno (2009*a*) studied group strategy-proof social choice functions with binary ranges and proved that strong group strategy-proofness implies group strategy-proofness, and the latter implies weak group strategy-proofness. This result holds under our extension of single-dipped preferences with indifferences, we show this in the Appendix. In a second paper, Barberá et. al. (2009*b*) proved that one condition is sufficient for a profile to have this equivalence between strategy-proofness and group strategy-proofness which is the *sequential inclusion* property. In fact, the profile of single-dipped preferences with indifferences satisfies such condition. In this paper we are going to hold to strong group strategy-proofness.

A different approach for the problem of locating a public facility is to consider the location of public good. Under this framework, each agent has an ideal point about where to locate the public good which is called the "peak". As we move away from the peak in each direction the agent's utility strictly decreases. This type of preferences is called "*single-peak preferences*". Moulin (1980) characterized the family of strategy-proof social choice functions which is known as the "generalized median voter rules". These rules are those which select the median of the n peaks plus $n-1$ phantom voters located along the alternative space. Furthermore, the generalized median voter rules are group strategy-proof and efficient. Cantala (2004) extended the concept of single-peak preferences to allow indifferences in the same sense that from one point onward the agent does not care anymore about the location of the public good. To illustrate such preferences, consider the following figure with 5 agents.

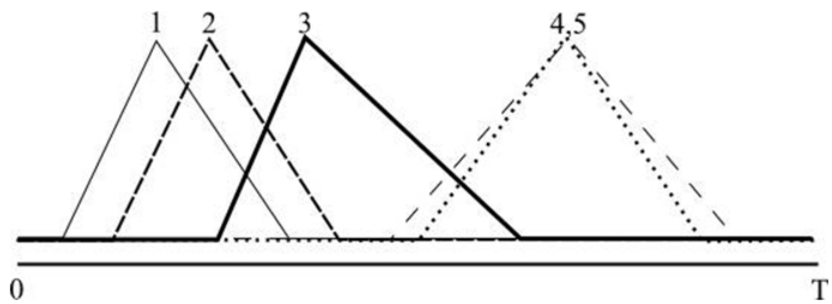

Figure 3. Single-peak preferences with indifferences.

The only strategy-proof, efficient, and anonymous social choice function is the median voter rule with the phantom voters located at the bounds of the alternative space. Hence, the only possible outcome under this framework is the selection of a peak of an agent, while the phantoms are not in the range of the function. That is, under single-peak preferences, the inclusion of indifferences reduce the range of the efficient and strategy-proof social choice function. In this paper, we show that exactly the contrary happens when indifferences are introduced in the context of single-dipped preferences.

The paper is organized as follows. In section 2 we introduce the basic notation and definitions. In section 3 we prove our main characterization result. Then, in section 4 we present some final remarks. Finally, in the Appendix we prove that strategy-proofness is equivalent to group strategy-proofnes in our setting, and we prove that our results about the range of the social choice function hold in a more general setting.

2 Notation and definitions

Let $N = \{1, ..., n\}$ be the set of agents and let $[0, T] \subseteq \mathbb{R}$ be the set of alternatives. The preference of each agent $i \in N$ is a complete, reflexive, continuous, and transitive binary relation R_i over $[0,T]$. We denote the strict part of R_i by P_i and the indifference part of R_i by I_i. Let $\mathcal{R}$ denote the class of all possible preferences on $[0,T]$. We assume that, for all $i \in N$, R_i is single-dipped with indifferences; i.e., there exist a unique "dip" $d(R_i)$ and $l(R_i), h(R_i) \in [0,T]$, $l(R_i) < h(R_i)$, such that:

for all $a, b \in [l(R_i), h(R_i)]$ such that $[a < b \leq d(R_i)$ or $d(R_i) \leq b < a]$, then aP_ib;
for all $a, b \in [0, l(R_i)]$ then aI_ib,
for all $a, b \in [h(R_i), T]$ then aI_ib, and
if $0 < l(R_i) < h(R_i) < T$ then $l(R_i)I_ih(R_i)$.

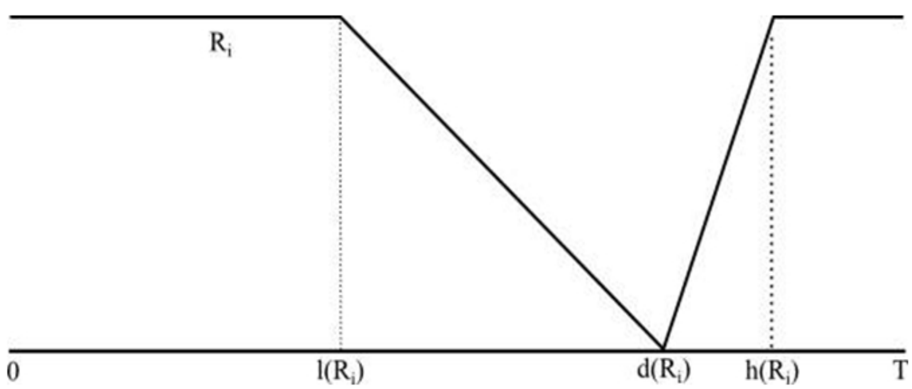

Figure 4. Single-dipped preferences with indifferences.

We assume that the indifferences appear when there exist a level of maximum satisfaction or a satiation level for each individual. Notice that we rule out cases like in figure 5.

FIGURE!!!

Figure 5. Preferences not included in the Single-dipped preferences with indifferences.

We use $\mathcal{R}_D$ to denote the class of all possible preferences that are single-dipped with indifferences on $[0,T]$.

A preference profile $R = (R_1, ..., R_n)$ is a $n-$tuple of all agents' preferences. Let $\mathcal{R}_D^n$ denote the class of all possible preference profiles where each agent's preferences is single-dipped with indifferences, i.e. $\mathcal{R}_D^n = \mathcal{R}_D \times ... \times \mathcal{R}_D$. Let $i \in N$ be an agent and R be a preference profile; denote by R_{-i} the $n-1$ tuple of all agents' preferences except i. For each $M \subseteq N$, R_M denotes all preferences of the agents in M, and R_{-M} denotes all preferences of the agents that are not in M.

Given $R \in \mathcal{R}_D^n$, let

$$\begin{aligned} N_0(R) &= \{i \in N : 0P_iT\}, \\ N_T(R) &= \{i \in N : TP_i0\}, \text{ and} \\ N_{0T}(R) &= \{i \in N : 0I_iT\}. \end{aligned}$$

Given $M \subseteq N$ and $R_M \in \mathcal{R}_D^m$, let

$$\begin{aligned} l_{min}(R_M) &= \min\{l(R_i) : i \in M\} \text{ and} \\ h_{max}(R_M) &= \max\{h(R_i) : i \in M\}. \end{aligned}$$

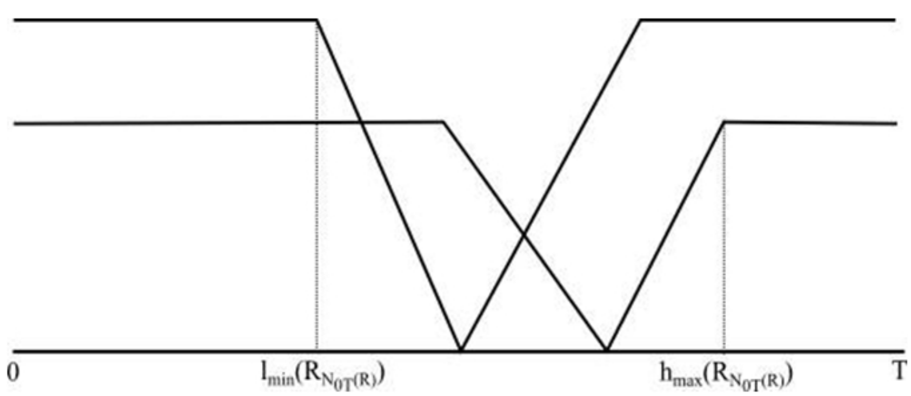

Figure 6. General setting of Single-dipped prefereces with indifferences.

Notice that if $N_0(R)$ and $N_T(R)$ are non empty, then $h_{max}(R_{N_0(R)}) = T$ and $l_{min}(R_{N_T(R)}) = 0$.

Definition 1 *For all $R_i \in \mathcal{R}_D$ and $a \in [0,T]$, we define the set of* ***worst alternatives*** *with respect to a given a preference R_i as $W(a; R_i) = \{b \in [0,T] : aR_ib\}$.*

Given $R \in \mathcal{R}_D^n$, let

$$a_0(R) = \min_{i \in N_T(R)} [\max W(0; R_i)] \text{ and}$$
$$a_T(R) = \max_{i \in N_0(R)} [\min W(T; R_i)].$$

Figure 7 illustrates the previous notations.

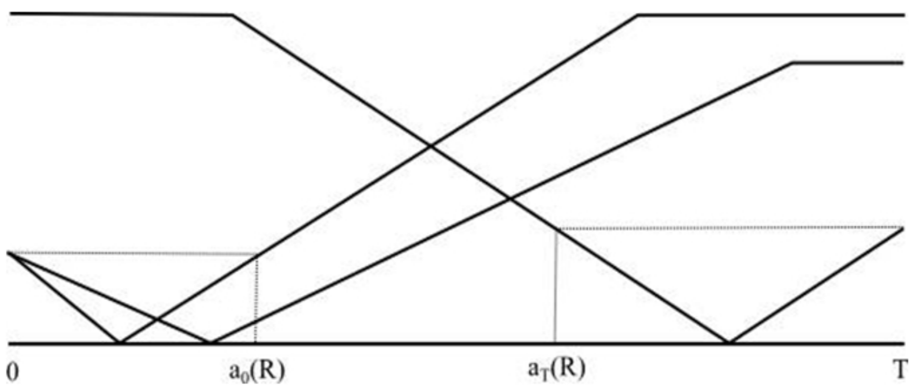

Figure 7. General setting of Single-dipped prefereces with indifferences.

A social choice function f associates each preference profile with an alternative, i.e. $f : \mathcal{R}_D^n \to [0, T]$. The social choice function f will add up each of the agent's preferences over the alternatives $[0, T]$ to decide a single alternative to place the public bad. This function can be as general as possible but we may ask to achieve some requirements to avoid misrepresentations of the preferences or to achieve a social outcome which satisfy a minimal condition of respect for individual preferences.

Definition 2 *Given $R \in \mathcal{R}_D^n$ the **Pareto efficient set** is $P(R) = \{a \in [0, T] : \nexists$ $b \in [0, T]$ such that bR_ia for all $i \in N$ and bP_ja for some $j \in N\}$.*

Definition 3 *Given $R \in \mathcal{R}_D^n$ and $A \subseteq [0, T]$ the **Pareto improvement set** at A is $E(A; R) = \{a \in P(R) :$ for all $b \in A$ we have that aR_ib for all $i \in N$ and aP_jb for some $j \in N\}$.*

Notice that $P(R)$ and $E(A; R)$ can be empty. Furthermore, $E(A; R) \subseteq P(R)$.

Example 1 *Let $n = 2$, and let $R \in \mathcal{R}_D^2$ be such that $0P_1T$ and TP_20 as in the following figure. Notice that $E(\{T\}; R) = h(R_2)$ and $E([\{0\}, a_0(R)]; R) = 0$. Moreover, $P(R) = \{0\} \cup [a_0(R), h(R_2)]$.*

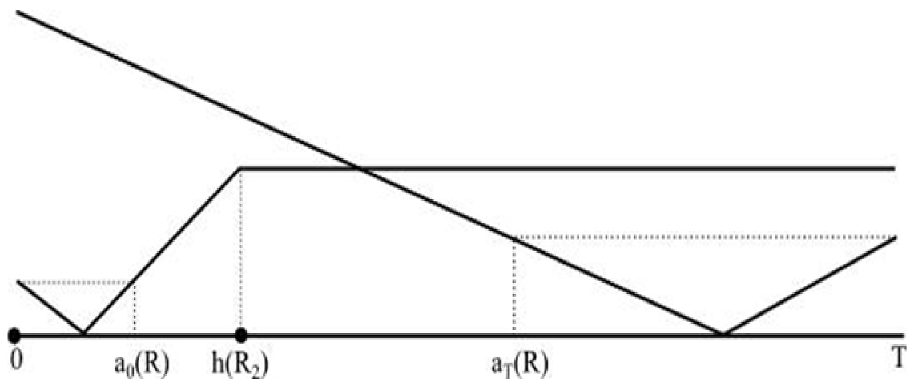

Figure 8. $E(\{T\}; R) = h(R_2)$ and $E([\{0\}, a_0(R)]; R) = 0$

Definition 4 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **Pareto efficient SCF** if for each $R \in \mathcal{R}_D^n$ such that $P(R) \neq \emptyset$, we have that*

$$f(R) \in P(R).$$

Definition 5 *Given $R \in \mathcal{R}_D^n$, a **unanimous interval** in R is an interval $[b, c] \subseteq [0, T]$ such that for each $a \in [b, c]$ and each $i \in N$, we have that:*

$$aR_id \text{ for each } d \in [0, T].$$

Given $R \in \mathcal{R}_D^n$, let $U(R)$ be the union of all the unanimous intervals in R.

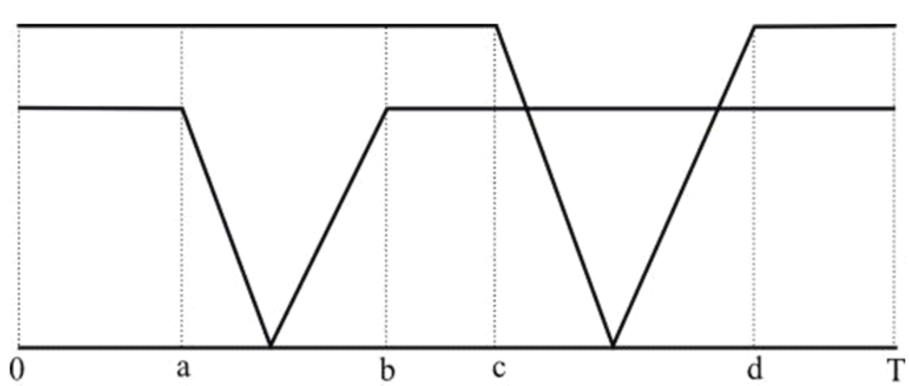

Figure 9. $U(R) = [0, a] \cup [b, c] \cup [d, T]$

Definition 6 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **unanimous** if for each $R \in \mathcal{R}_D^n$ such that $U(R) \neq \emptyset$, we have that*

$$f(R) \in U(R).$$

If a social choice function f is unanimous it will select an alternative considered as least as good as any other alternative in $[0, T]$ for all agents. Notice that Pareto efficiency is a stronger concept of efficiency than unanimity, hence, Pareto efficiency implies unanimity.

We have mentioned another appealing property commonly studied in the literature: strategy-proofness. If a social choice function is strategy-proof then an agent will not have any gain by submiting a different preference.

Definition 7 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **strategy-proof** if for each $i \in N$ and each $R \in \mathcal{R}_D^n$, there is not $R'_i \in \mathcal{R}$ such that*

$$f(R'_i, R_{-i}) P_i f(R).$$

Not only do we care about the misrepresentation of the preferences done by a single agent but also by a subset of them. There are different definitions of group manipulation of a social choice function. The difference between group strategy-proofness and strong group strategy-proofness is that in the first one the misrepresentation of the preferences done by a subset of agents has to lead to an outcome which is strictly preferred than the alternative elected under the true preference profile for every agent in the deviating subset. The latter condition asks that all the agents involved in the misrepresentation of the preferences remain as least as good with the new alternative as with the one chosen under the true preference profile and at least one of them is striclty better off.

Definition 8 *A social choice function $f : \mathcal{R}_D^n \to [0,T]$ is* ***group strategy-proof*** *if for each $M \subseteq N$ and each $R \in \mathcal{R}_D^n$, there is not $R'_M \in \mathcal{R}_D^m$, such that*

$$f(R'_M, R_{-M}) P_i f(R)$$

for all $i \in M$.

Definition 9 *A social choice function $f : \mathcal{R}_D^n \to [0,T]$ is* ***strong group strategy-proof*** *if for each $M \subseteq N$ and each $R \in \mathcal{R}_D^n$, there is not $R'_M \in \mathcal{R}_D^m$, such that*

$$f(R'_M, R_{-M}) R_i f(R)$$

for all $i \in M$, and $f(R'_M, R_{-M}) P_j f(R)$ for some $j \in M$.

In general, strong group strategy-proofness implies group strategy-proofness, and the latter implies strategy-proofness. We show in the appendix that under single-dipped preferences with indifferences strategy-proofness implies group strategy-proofness; however, strategy-proofness does not imply strong group strategy-proofness. To illustrate the case, we are going to consider the family of the serial dictatorship rules in the following example.

Example 2 *Let $\prec$ be a linear order on N such that $1^{\prec} \prec 2^{\prec} \prec ... \prec n^{\prec}$ represents the complete order of the n agents. A simple tie-braker $\hat{t}$ is any function $\hat{t} : \mathcal{R}_D^n \to \{0, T\}$. The* ***serial dictatorship rule*** *given the ordering $\prec$ and tie-breaker $\hat{t}$ is a social choice function $f^{(\prec,\hat{t})} : \mathcal{R}_D^n \to [0,T]$ such that for each $R \in \mathcal{R}_D^n$ we have that*

$$f^{(\prec,\hat{t})}(R) = \begin{cases} 0 & \text{if } 1^{\prec} \in N_0(R) \\ T & \text{if } 1^{\prec} \in N_T(R) \\ 0 & \text{if } 1^{\prec} \in N_{0T}(R) \text{ and } 2^{\prec} \in N_0(R) \\ T & \text{if } 1^{\prec} \in N_{0T}(R) \text{ and } 2^{\prec} \in N_0(R) \\ \vdots & \\ \hat{t}(R) & \text{if } N_{0T}(R) = N. \end{cases}$$

The serial dictatorship rule is unanimous and strategy-proof but it is not strong group strategy-proof. Consider the ordering $\prec$ such that $1^{\prec} \in N_{0T}(R)$, $2^{\prec} \in N_0(R)$, and

$3^{\prec} \in N_T(R)$, *hence* $f^{(\prec,\hat{t})}(R) = 0$. *The subset of agents* $M = \{1^{\prec}, 3^{\prec}\}$ *misrepresents their preferences by reporting any* $R'_M \in \mathcal{R}_D^2$ *such that* $1^{\prec} \in N_T(R'_M, R_{-M})$ *and* $3^{\prec} \in N_T(R'_M, R_{-M})$. *Then,* $f^{(\prec,\hat{t})}(R'_M, R_{-M}) = T$. *Hence* $f^{(\prec,\hat{t})}$ *is not strong group strategy-proof as* $TI_{1^{\prec}}0$ *and* $TP_{3^{\prec}}0$.

It is also important to notice that unanimity and strong group strategy-proofness imply efficiency. The reason for this is the following: if an alternative which is not efficient is chosen then all the individuals are going to misrepresent their preferences in order to choose an efficient alternative. Also notice that strong group strategy-proofness alone does not guarantee efficiency because a social choice function that always chooses the same alternative is strong group-strategy proof but not efficient.

In this section we presented the formal definitions and discussed some properties that a social choice function might achieve. In the following section we show the main results of the paper.

3 Results

We start this section presenting intermediate results that will help us prove the main theorem of the paper which is presented at the end of this section. The following lemmas describe the range of the strong group strategy-proof and unanimous social choice functions under single-dipped preferences with indifferences.

Lemma 1 *If a social choice function* $f : \mathcal{R}_D^n \to [0, T]$ *is strong group strategy-proof and unanimous then*

$$f(R) \notin (0, a_0(R)) \text{ and}$$
$$f(R) \notin (a_T(R), T).$$

Proof. Let $f : \mathcal{R}_D^n \to [0, T]$ be strong group strategy-proof and unanimous. Suppose that there exists $R \in \mathcal{R}_D^n$ such that $f(R) \in (0, a_0(R))$.

Notice that $f(R) \in W(0, R_i)$ for each $i \in R_{N_T(R)}$. Moreover, for each $i \in N_T(R)$ we have that $0 P_i f(R)$.

For each $i \in N_T(R)$, construct $R'_i \in \mathcal{R}_D$ such that $0 P'_i T$ and $l_{min}(R) = 0$. Notice, $N_T(R'_{N_T(R)}, R_{-N_T(R)}) = \emptyset$. Thus, $U(R'_{N_T(R)}, R_{-N_T(R)}) = 0$, and by unanimity $f(R') = 0$. This violates group strategy-proofness.

The proof is symmetric for the case where there exists a profile $R \in \mathcal{R}^n_D$ such that $a_T(R) \neq T$ and $f(R) \in (a_T(R), T)$. ■

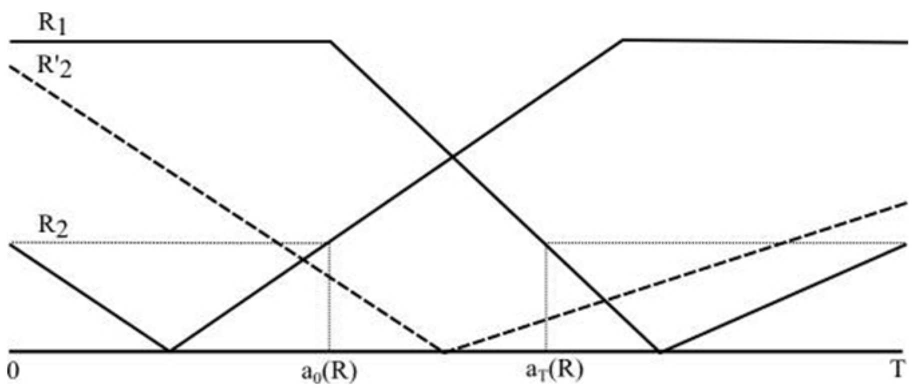

Figure 10. Let $N = \{1, 2\}$ such that $0 P_1 T$ and $T P_2 0$. If $f(R) \in (0, a_0(R))$, then agent 2 reports R'_2 such that $0 P_2 T$ and by unanimity $f(R') P_2 f(R)$.

Lemma 1 implies the following remarks.

Remark 1 *If a social choice function $f : \mathcal{R}^n_D \to [0, T]$ is strong group strategy-proof and unanimous for each $R \in \mathcal{R}^n_D$ such that $a_0(R) > a_T(R)$, then*

$$f(R) \notin (a_T(R), a_0(R)).$$

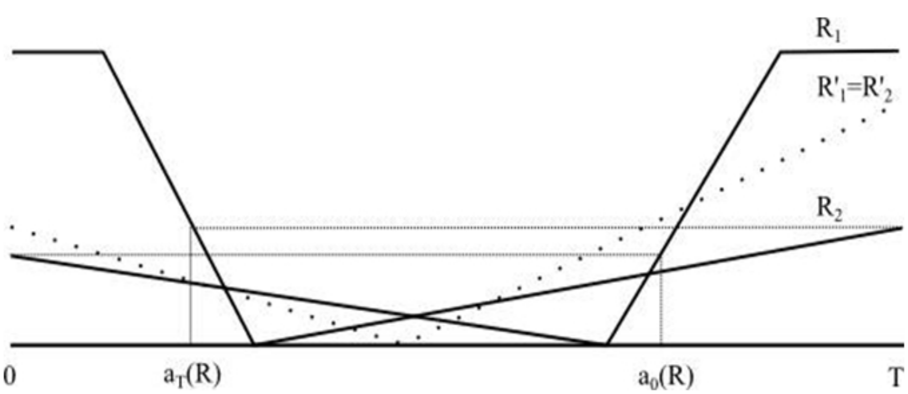

Figure 11. If $a_0(R) > a_T(R)$, then $f(R) \notin (a_T(R), a_0(R))$.

Remark 2 *Given a strong group strategy-proof and unanimous social choice function $f : \mathcal{R}_D^n \to [0,T]$ for all $R \in \mathcal{R}_D^n$ such that $a_0(R) > a_T(R)$, then*

$$f(R) \in \{0, T\}.$$

Lemma 2 *Given a strong group strategy-proof and unanimous social choice function $f : \mathcal{R}_D^n \to [0,T]$ for all $R, R' \in \mathcal{R}_D^n$ such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)} \in \mathcal{R}_D^{\#N_{0T}(R)}$, $a_0(R) > a_T(R)$, and $a_0(R') > a_T(R')$; then*

$$f(R) = 0 \text{ and } f(R') = 0, \text{ or}$$
$$f(R) = T \text{ and } f(R') = T.$$

Proof. Let f be strong group strategy-proof and unanimous, and let $R, R' \in \mathcal{R}_D^n$ be such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)} \in \mathcal{R}_D^{\#N_{0T}(R)}$, $a_0(R) > a_T(R)$, and $a_0(R') > a_T(R')$. By Remark 2 $f(R) \in \{0, T\}$ and $f(R') \in \{0, T\}$. Suppose that $f(R) = 0$. A symmetric argument applies if $f(R) = T$.

Suppose $(R'_{N_T(R)}, R_{-N_T(R)})$ is such that $N_0(R) = N_0(R'_{N_T(R)}, R_{-N_T(R)})$, $N_T(R) = N_T(R'_{N_T(R)}, R_{-N_T(R)})$, $R_{N_{0T}(R)} = R'_{N_{0T}(R'_{N_T(R)}, R_{-N_T(R)})} \in \mathcal{R}_D^{\#N_{0T}(R)}$, and $a_0(R'_{N_T(R)}, R_{-N_T(R)}) > a_T(R'_{N_T(R)}, R_{-N_T(R)})$. In order to obtain a contradiction, let $f(R'_{N_T(R)}, R_{-N_T(R)}) = T$. Notice that, $f(R'_{N_T(R)}, R_{-N_T(R)}) P_i f(R)$ for each $i \in N_T(R)$. This contradicts that f is group strategy-proof. ■

Now, let us consider the case where $a_0(R) \leq a_T(R)$.

Lemma 3 *Let $f : \mathcal{R}_D^n \to [0,T]$ be a strong group strategy-proof and unanimous social choice function for all $R \in \mathcal{R}_D^n$ such that $a_0(R) \leq a_T(R)$, $a_0(R) \neq 0$, and $a_T(R) \neq T$. If $f(R) \in [a_0(R), a_T(R)]$ then*

$$f(R) \in U(R) \cup E(\{0\}; R) \cup E(\{T\}; R)$$

Proof. Let f be strong group strategy-proof and unanimous, and let $R \in \mathcal{R}_D^n$ be such that $0 \neq a_0(R) \leq a_T(R) \neq T$, $f(R) \notin U(R)$, $f(R) \notin E([0]; R)$, $f(R) \notin E([T]; R)$ and $f(R) \in [a_0(R), a_T(R)]$.

Now construct a $R' \in \mathcal{R}_D^n$ such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)}$, and $a_0(R') > a_T(R')$. Assume that $f(R') = 0$.

For all $i \in N_0(R) = M$, let $R''_i \in \mathcal{R}_D$ be such that $N_0(R''_M, R_{-M}) = N_0(R')$, $N_T(R''_M, R_{-M}) = N_T(R')$, $R''_{N_{0T}(R)} = R'_{N_{0T}(R')}$, and $a_0(R''_M, R_{-M}) > a_T(R''_M, R_{-M})$. Then by Lemma 2, $f(R''_M, R_{-M}) = 0$. Notice that, $f(R''_M, R_{-M}) R_i f(R)$ for each $i \in M$.

Since $f(R) \notin U(R)$ and $f(R) \notin E([0]; R)$ then we have two cases:

(i) There exists $i \in M$ such that $f(R''_M, R_{-M}) P_i f(R)$. This violates strong group strategy-proofness.

(ii) If there is no $i \in M$ such that $f(R''_M, R_{-M}) P_i f(R)$, this implies that $f(R) \in [a_0(R), l_{min}(R_{N_0(R)})]$. Then there exists $i^* \in N_T(R)$ such that $f(R) P_{i^*} f(R''_M, R_{-M})$. Then the group $\{i^*\} \cup M$ violates strong group strategy-proofness.

The symmetric argument is used for the case $f(R') = T$. ■

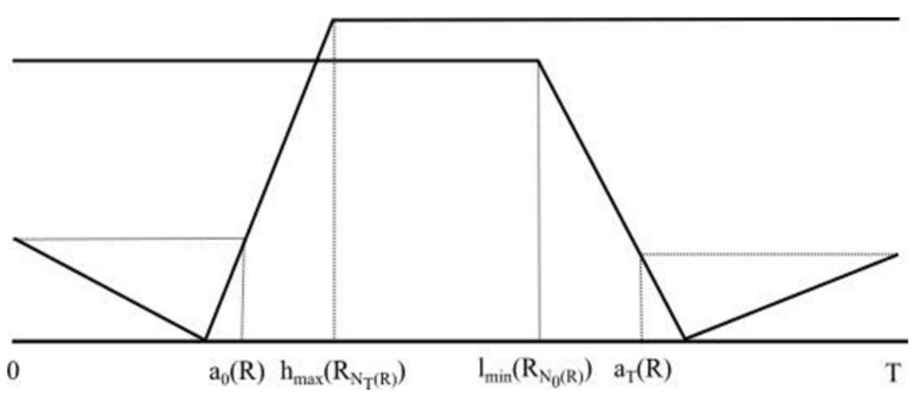

Figure 12. $U(R) \subseteq (a_0(R), a_T(R))$, then $f(R) \in [h_{max}(R_{N_T(R)}), l_{min}(R_{N_0(R)})] = U(R)$.

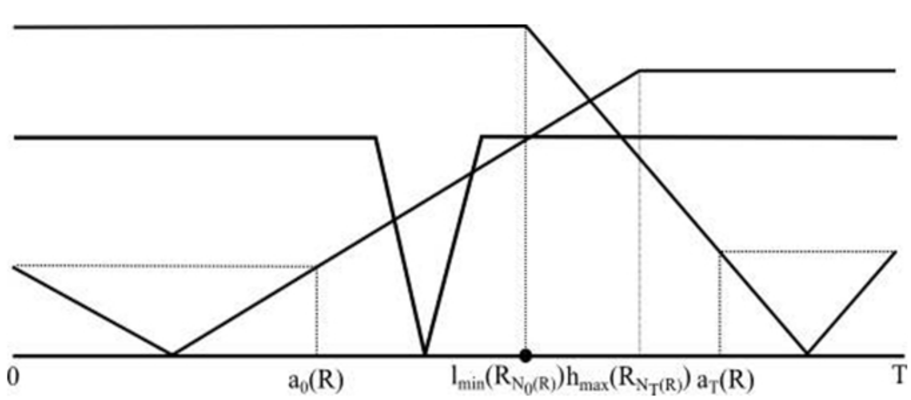

Figure 13. $E(\{0\};R) \neq \emptyset$, then $f(R) = l_{min}(R_{N_0(R)})$

Lemma 4 *Let $f : \mathcal{R}_D^n \to [0,T]$ be a strong group strategy-proof and unanimous social choice function for all $R \in \mathcal{R}_D^n$ such that $a_0(R) \leq a_T(R)$ and $[a_0(R) = 0$ or $a_T(R) = T]$. If $f(R) \in [a_0(R), a_T(R)]$ then*

$$f(R) \in U(R) \cup E(\{0\};R) \cup E(\{T\};R) \cup \{0\} \cup \{T\}.$$

Proof. Let f be strong group strategy-proof and unanimous, and let $R \in \mathcal{R}_D^n$ be such that $0 = a_0(R) \leq a_T(R)$, $f(R) \notin U(R)$, $f(R) \notin E([0];R)$, $f(R) \notin E([T];R)$, $f(R) \neq 0$, $f(R) \neq T$, and $f(R) \in [a_0(R), a_T(R)]$. Consider the following two cases:

(i) Let $a_T(R) \neq T$.

For all $i \in N_T(R)$, let $R'_i \in \mathcal{R}_D$ be such that $a_0(R'_{N_T(R)}, R_{-N_T(R)}) \neq 0$, $N_0(R) = N_0(R'_{N_T(R)}, R_{-N_T(R)})$, and $f(R'_{N_T(R)}, R_{-N_T(R)}) = T$. Notice that $R'_{N_T(R)}$ always exists since by Lemma 2 and Lemma 3 $f(R'_{N_T(R)}, R_{-N_T(R)}) \in \{0,T\}$; and if it does not exist then consider $R''_i \in \mathcal{R}_D$ for all $i \in N_T(R)$ such that $f(R)P_i f(R''_{N_T(R)}, R_{-N_T(R)}) = 0$, then the group $N_T(R)$ can manipulate going from $(R''_{N_T(R)}, R_{-N_T(R)})$ to R.

Since $f(R) \notin U(R)$ and $f(R) \notin E(\{T\};R)$ then we have two cases:

(a) There exists $i \in N_T(R)$ such that $f(R'_{N_T(R)}, R_{-N_T(R)})P_i f(R)$. This violates strong group strategy-proofness.

(b) If there is no $i \in N_T(R)$ such that $f(R'_{N_T(R)}, R_{-N_T(R)})P_i f(R)$, this implies that $f(R) \in [h_{max}(R_{N_T(R)}),\ a_T(R)]$. Then there exists $i^* \in N_0(R) \cup N_{0T}(R)$ such that $f(R)P_{i^*} f(R'_{N_T(R)}, R_{-N_T(R)})$. Then the group $\{i^*\} \cup M$ violates strong group strategy-proofness.

(ii) $a_T(R) = T$.

Case (i) implies that there is $R''' \in \mathcal{R}_D^n$ such that $a_0(R''') \leq a_T(R''')$, $[a_0(R''') \neq 0$ or $a_T(R''') \neq T]$, and $f(R''') \in \{0,T\}$. Then a similar argument than the one used in case (i) can be used here in order to obtain a contradiction.

The symmetric argument is used for the case $a_0(R) \neq 0$. ■

Lemmas 1 to 4 imply the following remark.

Remark 3 *If a social choice function* $f : \mathcal{R}_D^n \to [0,T]$ *is strong group strategy-proof and unanimous then,*

$$f(R) \in \begin{cases} U(R) & \text{if } U(R) \neq \emptyset \\ \{0\} \cup \{T\} \ \cup E(\{0\};R) \cup E(\{T\};R) & \text{otherwise} \end{cases}$$

Since strong group-strategy proofness and unanimity imply efficiency, notice that Remark 3 implies that if $U(R) = \emptyset$ then the election is between two options. Since if $E(\{0\};R) \neq \emptyset$ then 0 is not going to be elected and/or if $E(\{T\};R) \neq \emptyset$ then T is not going to be elected. Therefore, for each $R \in \mathcal{R}_D^n$ such that $U(R) = \emptyset$ we have that only one of two alternatives can be chosen.

By previous results in the literature, we know that whenever the choice is only between two alternatives then strategy-proof rules can only be described as choosing one alternative unless there is enough support for the opposite in which case the other alternative is chosen (See Barberà (2010) for a review of these results). Therefore, we are going to concentrate on this type of rules, since a subset of this kind of rules are strong group strategy-proof.

Before we continue with the results, we are going to present a family of rules that satisfies strong group strategy-proof and unanimity. Moreover, we are going to show that this is the unique family satisfying both properties when $n \neq 2$.

Let $A \subseteq [0,T]$ be a non-empty set of alternatives. Define a tie-breaker $t = \{t[A]\}_{A \subseteq [0,T]}$ as the family of tie-breaker functions which associates to each preference profile R an alternative in $A \subseteq [0,T]$; i.e. $t[A] : \mathcal{R}_D^n \to A$.

Definition 10 *Let* $b \in \{0,T\}$ *be a bias and* t *a tie-breaker. For each* $R \in \mathcal{R}_D^n$ *we define a* ***full agreement rule*** $f^{(b,t)}$ *with bias* b *and tie-breaker* t *as:*

$$f^{(b,t)}(R) = \begin{cases} t[U(R)] & \text{if } U(R) \neq \emptyset \\ t[E([a_0(R), l_{min}(R_{N_0(R)}); R)] & \text{if } U(R) = \emptyset,\ b = 0 \text{ and } E(\{0\};R) \neq \emptyset \\ 0 & \text{if } U(R) = \emptyset,\ b = 0 \text{ and } E(\{0\};R) = \emptyset \\ t[E([h_{max}(R_{N_T(R)}), a_T(R); R)] & \text{if } U(R) = \emptyset,\ b = T \text{ and } E(\{T\};R) \neq \emptyset \\ T & \text{if } U(R) = \emptyset,\ b = T \text{ and } E(\{T\};R) = \emptyset \end{cases}$$

Notice that, when $b = 0$ then T is chosen if $U(R) = \{T\}$ otherwise 0 is chosen (assuming that $E(\{0\}; R) = \emptyset$); and when $b = T$ then 0 is chosen if $U(R) = \{0\}$ otherwise T is chosen (assuming that $E(\{T\}; R) = \emptyset$). Thus, this is as asking all the support for T in order to be chosen when $b = 0$ otherwise 0 is elected, and all the way around when $b = T$.

In order to simplify our argument, let us consider those cases where $U(R) = \emptyset$, $E(\{0\}; R) = \emptyset$ and $E(\{T\}; R) = \emptyset$. Then, the only thing that we need to show in order to prove that the family of full agreement rules is the unique family of social choice functions satisfying strong group strategy-proofness and unanimity is that there is no other way to define a support for 0 or T different from the way done by the family of full agreement rules. In order to show that this proposition is true, consider the following cases:

(a) $n = 2$. In this case we can define a support for 0 or T as a function of the names of the individuals. For example, a rule can be defined as choosing 0 if agent 1 prefers 0 than T. This rule is strong group strategy-proof and unanimous. Since b is exogenous in the sense that do not depend on the set N, then the family of full agreement rules is not unique when $n = 2$.

(b) $n = 3$. Consider any $R \in \mathcal{R}_D^n$ such that $\#N_{0T}(R) = 1$. In this case, the agent $i \in N_{0T}(R)$ can manipulate if he is one of the individuals required to support T (0) in order for T (0) to be chosen, otherwise 0 (T) is chosen, unless we ask for: (i) all the individuals to support T (or 0), or (ii) only one individual to support T (or 0). Notice that this is the full agreement rule when $b = 0$ ($b = T$) in case (i), and $b = T$ ($b = 0$) in case (ii).

(c) $n > 3$. The same argument presented for the previous case (b) applies here.

Given our previous results we can state our main Theorem.

Theorem 1 *When $n \neq 2$, a social choice function $f : \mathcal{R}_D^n \to [0, T]$ is strong group strategy-proof and unanimous if and only if there exist a bias $b \in \{0, T\}$ and a tie-breaker t such that f is a full agreement rule $f^{(b,t)}$.*

4 Final Remarks

DESARROLLAR....

Remark 3 describe the range of a strong group strategy-proof and unanimous social choice function under single-dipped preferences with indifferences. As we can see, the cardinality of the range can be greater than two, opposite to the case where preferences are single-dipped.

....

Our characterization of the range of the strong group strategy-proof and unanimous social choice functions under single dipped preferences with indifferences, we can see that these rules depend only on the "tops" of the preferences; that is, the most prefered alternatives for each agent. This property is known in the literature as "top onliness", Barberá et.al. (1991), this property is generally satisfied by strategy-proof social choice functions.

5 Appendix

5.1 Strategy-proofness and group strategy-proofness

When agents have single-dipped preferences with indifferences strategy-proofness implies group strategy-proofness.

Proposition 1 *If a social choice function $f : \mathcal{R}_D^n \to [0,T]$ is strategy-proof then f is group strategy-proof.*

Proof. As an induction hypothesis suppose that Proposition 1 holds for groups of l agents, then we are going to show that this proposition is true for groups of $l+1$ agents.

In order to obtain a contradiction, suppose that there is a group $M \subseteq N$ of $l+1$ agents and $R, R' \in \mathcal{R}_D^n$ such that $f(R'_M, R_{-M}) P_i f(R)$, for each $i \in M$.

By strategy-proofness we have that $f(R'_i, R_{-i}) \in W(f(R); R_i)$, for each $i \in M$.

By the induction hypothesis, for each $i \in M$ there exists $j_i \in M \setminus \{i\}$ such that $f(R'_i, R_{-i}) R_{j_i}\ f(R'_M, R_{-M})$; since if this not happens there is $i \in M$ and $j_i \in M \setminus \{i\}$ such that $f(R'_M, R_{-M}) P_{j_i} f(R'_i, R_{-i})$, and notice that the group $M \setminus \{i\}$ violates group strategy-proofness if the preference profile is (R'_i, R_{-i}).

Now, suppose that $f(R'_M, R_{-M}) < f(R)$. A symmetric argument applies when $f(R'_M, R_{-M}) > f(R)$.

There are only two possible cases:

(i) For each $i \in M$ and each $a \leq f(R)$, $a R_i f(R)$ (this means that $d(R_i) \geq f(R)$ for all $i \in M$).

Let $m \in \arg\min_{q \in M} \max[W(f(R); R_q)]$. Notice that, $W(f(R); R_m) \subseteq W(f(R); R_q)$ for all $q \in M$.

Then, for each agent $j \in M \setminus \{m\}$ we have that $f(R'_m, R_{-m}) \in W(f(R); R_j)$, since by strategy-proofness $f(R'_m, R_{-m}) \in W(f(R); R_m)$. In particular, this is true for j_i, then $f(R) R_{j_i}\ f(R'_i, R_{-i})$.

Finally, since $f(R'_i, R_{-i}) R_{j_i} f(R'_M, R_{-M})$ and by transitivity, we have that $f(R) R_{j_i} f(R'_M, R_{-M})$. This is in contradiction with $f(R'_M, R_{-M}) P_i f(R)$, for each $i \in M$.

(ii) There exists an agent $i \in M$ and $a < f(R)$, such that $f(R) P_i a$.

In order to obtain a contradiction, let i be such agent. By strategy-proofness, $f(R'_i, R_{-i}) \in W(f(R); R_i)$, then $f(R'_i, R_{-i}) < f(R)$. Notice that, $f(R'_i, R_{-i})$ is in between $f(R'_M, R_{-M})$ and $f(R)$, since $f(R'_M, R_{-M}) \notin W(f(R); R_i)$ which implies $f(R'_M, R_{-M}) < f(R'_i, R_{-i})$. Then by single-dippedness with indifferences and by the fact that $f(R'_i, R_{-i}) R_{j_i} f(R'_M, R_{-M})$, we have that $f(R) R_{j_i} f(R'_i, R_{-i})$. Again we obtain a contradiction, using transitivity and that $f(R'_i, R_{-i}) R_{j_i} f(R'_M, R_{-M})$. ■

5.2 Generalized single-dipped preferences with indifferences

In this section we are going to show that similar results about the range of the function hold under a generalization of the assumption made on preferences. From now on consider that for all $i \in N$, $\hat{R}_i$ is **generalized single-dipped preferences with indifferences** when there exist $l(\hat{R}_i), h(\hat{R}_i) \in [0,T]$, $l(\hat{R}_i) < h(\hat{R}_i)$, and a unique "dip" $d(\hat{R}_i)$ such that:

for all $a, b \in [l(\hat{R}_i), h(\hat{R}_i)]$ we have that $[a < b \leq d(\hat{R}_i)$ or $d(\hat{R}_i) \leq b < a]$ implies that $a\hat{P}_i b$,
for all $a, b \in [0, l(\hat{R}_i)]$ we have that $aI_i b$, and
for all $a, b \in [h(\hat{R}_i), T]$ we have that $aI_i b$.

In this case, the range of the strong group strategy-proof and unanimous social choice function is wider since $l_{min}(\hat{R}_{N_T(\hat{R})}) \geq 0$ and $h_{max}(\hat{R}_{N_0(\hat{R})}) \leq T$. In particular,

$$f(\hat{R}) \in \begin{cases} U(\hat{R}) & \text{if } U(\hat{R}) \neq \emptyset \\ [0, l_{min}(\hat{R})] \cup [h_{max}(\hat{R}), T] \ \cup E([0]; \hat{R}) \cup E([T]; \hat{R}) & \text{otherwise} \end{cases}$$

In order to obtain this result, we add the following lemmas which will complete our proof.

Lemma 5 *Given a strong group strategy-proof and unanimous social choice function $f : \hat{\mathcal{R}}_D^n \to [0,T]$ for all $\hat{R} \in \hat{\mathcal{R}}_D^n$ such that $a_0(\hat{R}) > a_T(\hat{R})$, then*

$$f(\hat{R}) \in [0, l_{min}(\hat{R})] \cup [h_{max}(\hat{R}), T].$$

Proof. Assume that f is strong group strategy-proof and unanimous, and there exists $\hat{R} \in_D^n$ such that $a_0(\hat{R}) > a_T(\hat{R})$, and $f(\hat{R}) \notin [0, l_{min}(\hat{R})] \cup [h_{max}(\hat{R}), T]$. By Remark 1, $f(\hat{R}) \notin (a_T(\hat{R}), a_0(\hat{R}))$.

Let $f(\hat{R}) \in (l_{min}(\hat{R}), a_T(\hat{R})]$. Then $0\hat{R}_i \ f(\hat{R})$ for all $i \in N$, and $0\hat{P}_{j^*} \ f(\hat{R})$ for some $j^* \in N$ such that $l(\hat{R}_{j^*}) < f(\hat{R})$. Now consider the $\hat{R}' \in \hat{\mathcal{R}}_D^n$ such that for each $a \in (0,T]$ we have that $0\hat{P}_i a$, for all $i \in N$. By unanimity $f(\hat{R}') = 0$. Notice that $0\hat{R}_i f(\hat{R})$ for all $i \in N$ and $0\hat{P}_{j^*} \ f(\hat{R})$; this violates strong group strategy-proof. Contradiction.

The proof is symmetric for the case where $f(\hat{R}) \in [a_0(\hat{R}), h_{max}(\hat{R}))$. ■

Lemma 6 *If a social choice function* $f : \hat{\mathcal{R}}_D^n \to [0,T]$ *is strong group strategy-proof and unanimous for each* $\hat{R} \in \hat{\mathcal{R}}_D^n$ *such that* $l_{min}(\hat{R}_{N_T(\hat{R})}) \geq h_{max}(\hat{R}_{N_0(\hat{R})})$*, then*

$$f(\hat{R}) \notin [h_{max}(\hat{R}_{N_0(\hat{R})}), l_{min}(\hat{R}_{N_T(\hat{R})})].$$

Proof. Let $f : \hat{\mathcal{R}}_D^n \to [0,T]$ be strong group strategy-proof and unanimous. Suppose that there exist $\hat{R} \in \hat{\mathcal{R}}_D^n$ such that $f(\hat{R}) \in [h_{max}(\hat{R}_{N_0(\hat{R})}), l_{min}(\hat{R}_{N_T(\hat{R})})]$. See figure 15.

Notice that, for each $i \in N$ and for each $a \in [0, l_{min}(\hat{R})]$, $a\hat{R}_i f(\hat{R})$. Moreover, for each $i \in N_0(\hat{R})$ and for each $a \in [0, l_{min}(\hat{R})]$, $a\hat{P}_i f(\hat{R})$.

For each $i \in N_T(\hat{R})$, construct $\hat{R}' \in \hat{\mathcal{R}}_D^n$ such that $0\hat{P}_i'T$, and $l_{min}(\hat{R}'_{N_T(\hat{R})}, \hat{R}_{-N_T(\hat{R})}) = l_{min}(\hat{R})$. Notice, $N_T(\hat{R}'_{N_T(\hat{R})}, \hat{R}_{-N_T(\hat{R})}) = \emptyset$. Thus, $U(\hat{R}'_{N_T(\hat{R})}, \hat{R}_{-N_T(\hat{R})}) = [0, l_{min}(\hat{R})]$, and by unanimity $f(\hat{R}') \in [0, l_{min}(\hat{R})]$. This violates strong group strategy-proof. ■

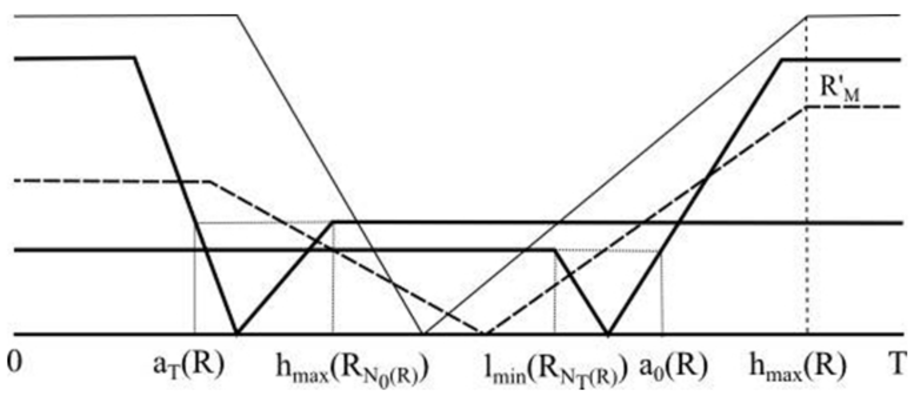

Figure 15. $l_{min}(\hat{R}_{N_T(\hat{R})}) \geq h_{max}(\hat{R}_{N_0(\hat{R})})$

Lemma 7 *Given a strong group strategy-proof and unanimous social choice function* $f : \hat{\mathcal{R}}_D^n \to [0,T]$ *for all* $\hat{R}, \hat{R}' \in \hat{\mathcal{R}}_D^n$ *such that* $N_0(\hat{R}) = N_0(\hat{R}')$*,* $N_T(\hat{R}) = N_T(\hat{R}')$*,* $\hat{R}_{N_{0T}(\hat{R})} = \hat{R}'_{N_{0T}(\hat{R})} \in \hat{\mathcal{R}}_D^{\#N_{0T}(\hat{R})}$*,* $l_{min}(\hat{R}) = l_{min}(\hat{R}')$*,* $h_{max}(\hat{R}) = h_{max}(\hat{R}')$*,* $a_0(\hat{R}) > a_T(\hat{R})$*, and* $a_0(\hat{R}') > a_T(\hat{R}')$*; then*

$$f(\hat{R}) \in [0, l_{min}(\hat{R})] \text{ and } f(\hat{R}') \in [0, l_{min}(\hat{R}')], \text{ or}$$
$$f(\hat{R}) \in [h_{max}(\hat{R}), T] \text{ and } f(\hat{R}') \in [h_{max}(\hat{R}'), T].$$

The proof is similar to the one presented for the case of single-dipped preferences with indifferences.

Lemma 8 *If a social choice function $f : \hat{\mathcal{R}}_D^n \to [0,T]$ is strong group strategy-proof and unanimous for all $\hat{R} \in \hat{\mathcal{R}}_D^n$ such that $a_0(\hat{R}) \leq a_T(\hat{R})$, then*

$$f(\hat{R}) \notin (l_{min}(\hat{R}), l_{min}(\hat{R}_{N_T(\hat{R})})] \cup [h_{max}(\hat{R}_{N_0(\hat{R})}), h_{max}(\hat{R})).$$

Proof. Assume that f is strong group strategy-proof and unanimous, and there exists $\hat{R} \in \hat{\mathcal{R}}_D^n$ such that $a_0(\hat{R}) \leq a_T(\hat{R})$ and $f(\hat{R}) \in (l_{min}(\hat{R}), l_{min}(\hat{R}_{N_T(\hat{R})})]$. Then $0\hat{R}_i\ f(\hat{R})$ for all $i \in N$, and $0\hat{P}_{j^*}\ f(\hat{R})$ for some $j^* \in N$ such that $l(\hat{R}_{j^*}) < f(\hat{R})$. Now consider the $\hat{R}' \in \hat{\mathcal{R}}_D^n$ such that for each $a \in (0,T]$ we have that $0\hat{P}_i a$, for all $i \in N$. By unanimity $f(\hat{R}') = 0$. Notice that $0\hat{R}_i f(\hat{R})$ for all $i \in N$ and $0\hat{P}_{j^*}\ f(\hat{R})$; this violates strong group strategy-proof. Contradiction.

For the case $f(\hat{R}) \in [h_{max}(\hat{R}_{N_0(\hat{R})}), h_{max}(\hat{R}))$ the proof works all the way around. ■

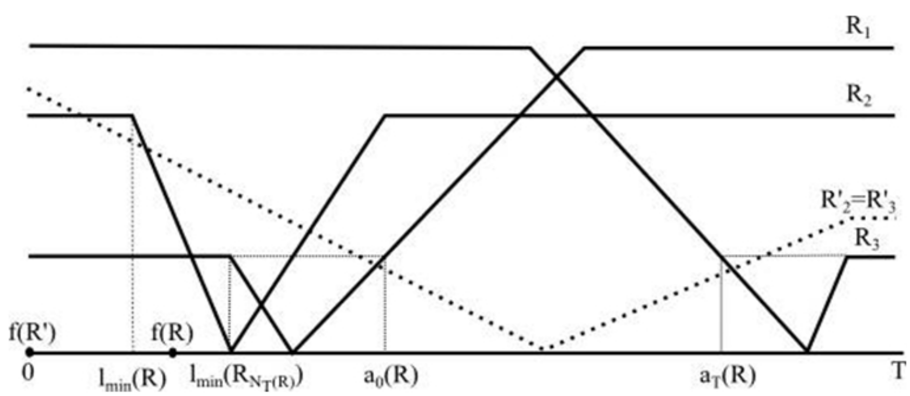

Figure 16. $f(\hat{R}) \notin (l_{min}(\hat{R}), l_{min}(\hat{R}_{N_T(\hat{R})})]$

References

[1] Barberá, S. (2007): "Indifferences and Domain Restrictions", *Analyse & Kritik*, 146–162

[2] Barberá, S., Berga, D., and Moreno, B. (2009*a*): "Single-dipped Preferences". Mimeo.

[3] Barberá, S., Berga, D., and Moreno, B. (2009*b*): "Individual versus Group Strategy-Proofness: When do They Coincide?". *Barcelona Working Paper Series*, Working paper 372

[4] Barberá, S. (2010): "Strategy-proof Social Choice", in K. J. Arrow, A. K. Sen and K. Suzumura (eds.), Handbook of Social Choice and Welfare, Volume 2, Chapter 25. North-Holland:Amsterdam.

[5] Barberá, S., Berga, D., and Moreno, B. (2010): "Group Strategy-proof Social Choice Functions with Binary Ranges and Arbitrary Domains: Characterization Results". Mimeo.

[6] Barberá, S., Sonnenschein, H., and Zhou, L. (1991): "Voting by Committeess", *Econometrica* $59 : 595 - 609$

[7] Cantala, D. (2004): "Choosing the Level of a Public Good when Agents Have an Outside Option", *Social Choice and Welfare* $22 : 491 - 514$

[8] Manjunath, V. (2009): "Efficient and Strategy-proof Social Choice when Preferences are Single-dipped". Mimeo.

[9] Manjunath, V. (2009): "A Note on Group Strategy-proofness and Social Choice with Two Alternatives". Mimeo.

[10] Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness", *Public Choice* $35 : 437 - 55$