ON THE PRIVATE PROVISION OF PUBLIC GOODS ON NETWORKS

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ABSTRACT. This paper analyzes the private provision of public goods where consumers interact within a fixed network structure and may benefit only from their direct neighbors' provisions. We present a proof for existence and uniqueness of a Nash equilibrium with general best-reply functions. Our uniqueness result simultaneously extends similar results in Bergstrom, Blume, and Varian (1986) on the private provision of public goods to networks and Bramoullé, Kranton, and D'Amours (2011) on games of strategic substitutes to nonlinear best-reply functions. In addition, we investigate the neutrality result of Warr (1983) and Bergstrom, Blume, and Varian (1986) whereby consumers are able to offset income redistributions and tax-financed government contributions. To this effect, we establish that the neutrality result has a limited scope of application beyond regular networks.

JEL classification: C72, D31, H41.

Keywords: public goods, uniqueness of Nash equilibrium, network games, neutrality, Bonacich centrality, main eigenvalue.

I am deeply grateful to Sanjeev Goyal for important and enlightening conversations in the early stages of this research. I also thank Ted Bergstrom, Yann Bramoullé, Richard Cornes, Roger Hartley, Bryony Reich, Oscar Volij, and seminar participants at SAET 2011 in Faro, PET 2011 meetings in Exeter and San Luis, CTN 2012 workshop in Paris, Southampton, Marseille (GREQAM), and Paris (CES) for helpful comments. The hospitality of the Economics Department at Cambridge University is gratefully acknowledged.

1. Introduction

The private provision of public goods is a subject of ongoing interest in several important strands of the economics literature ranging from taxation to political economy. Private contributions to public goods are important phenomena for many reasons. Voluntary contributions by members of a community are vital for the provision of essential social infrastructure, whilst at the aggregate level charitable giving accounts for a significant proportion of GDP in many countries. The seminal contribution of Bergstrom, Blume, and Varian (1986), built on an earlier striking result by Warr (1983), provides a rigorous investigation of the standard model of private provision of public goods.[1] Their main results, with sharp testable implications, are the neutrality of both aggregate public good provision and private good consumption to income redistribution among contributors and the complete crowding out of government contributions financed by lump-sum taxes preserving the set of contributors.

The findings of the private provision model rest on the assumption that each consumer benefits from the public good provisions of all other consumers. Often, for various public goods such as information gathering and new products experimentation, a consumer may benefit from provisions accessible only through his social interactions or geographical position. For instance, there is strong empirical evidence that farmers perceive the experimentation of a new technology as a public good and adjust their experimentation level in the opposite direction to their neighbors' provision (see, for example, Conley and Udry (2010)). Moreover, much consumption is a social activity and consumers often first seek information from friends, colleagues, or even their various online communities before sampling the products themselves.

In this paper, we investigate the private provision of public goods where consumers interact within a fixed network structure and benefit only from their direct neighbors' provisions. Recently, the economics of networks has gained prominence as a new approach to understand some of the patterns governing various economic interactions (see Goyal (2007) and Jackson (2008)). The main insights on formation and stability of networks are powerful predictive tools to both positive and normative analysis in many fields, including development economics and labor economics. Public goods provision on networks was first studied by Bramoullé and Kranton (2007). Their analysis, under complete information, distinguishes between specialized and hybrid contribution equilibria and shows that specialized contribution equilibria correspond to the maximal independent sets of the network. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) show that the possibility that consumers hold partial

[1] There is a special issue in the *Journal of Public Economics* celebrating the 20$^{\text{th}}$ anniversary of Bergstrom, Blume, and Varian (1986).

information about the network can shrink considerably the potentially large set of equilibria that arise under complete information. Galeotti and Goyal (2010) study a model of information sharing on a network where consumers simultaneously decide on their information provision and connections.

Bramoullé, Kranton, and D'Amours (2011) introduce a new approach to investigate games of strategic substitutes on networks[2] with linear best-reply functions. The main contribution is to introduce a new network measure related to the lowest eigenvalue of the adjacency matrix,[3] which is a key for equilibrium analysis. At the heart of their equilibrium analysis (existence requires a straightforward application of Brouwer's fixed point theorem), as in that of Bergstrom, Blume, and Varian (1986), lies the proof of uniqueness of the Nash equilibrium. Bergstrom, Blume, and Varian (1986) rely on the weak assumption of normality of private and public goods.[4] On the other hand, Bramoullé, Kranton, and D'Amours (2011) place a bound on the slope of the linear best-reply functions that relies on the lowest eigenvalue to establish the uniqueness of a Nash equilibrium. The proof technique appeals to the theory of potential games where consumers' optimal strategies concur in a common maximization problem of a potential function of which the strict concavity provides the uniqueness of a Nash equilibrium.

In this paper, we present a general proof of existence and uniqueness of a Nash equilibrium in the private provision of a public good on networks. We show that the shared ground of Bergstrom, Blume, and Varian (1986) and Bramoullé, Kranton, and D'Amours (2011) is beyond the trivial case of a complete network with linear best-reply functions. Indeed, our existence and uniqueness results simultaneously

[2]The private provision of public goods falls into this category since a consumer has incentives to adjust his public good provision in the opposite direction of his neighbors' provisions.

[3]Such a measure has not been used previously in any of the fields related to networks, including social networks, biology, and physics. Moreover, Bramoullé, Kranton, and D'Amours (2011) provide an interesting discussion on the structural properties of the network that may affect the lowest eigenvalue.

[4]However, the many subtleties of the proof may not have fully revealed the intuition behind the proof or shown what a familiar uniqueness argument is at work. For discussions and alternative proofs, see, for example, Bergstrom, Blume, and Varian (1992), Fraser (1992), and Cornes and Hartley (2007).

extend similar results in Bergstrom, Blume, and Varian (1986) on the private provision to a network setting and in Bramoullé, Kranton, and D'Amours (2011) on games of strategic substitutes to nonlinear best-reply functions. A crucial innovation of this paper is the uniqueness proof technique, which is based on an adaptation of Stiemke's Lemma to the private provision of public goods.[5] In our approach, we overcome the lack of linear structure by resorting to a network-specific normality assumption of both public and private goods which stipulates bounds on the nonlinear best-reply functions. In addition, an inherent advantage of our proof technique is that it applies directly to the original public good game and, therefore, it provides insights on what is driving the uniqueness result in this class of games.

The closely related literature on clubs/local public goods also investigates the strategic interactions underlying the formation of clubs and communities. If one thinks of a network as a collection of clubs formed either by the edges or the nodes then the public goods network literature and the club/local public goods literature are essentially equivalent. However, such an equivalence is not very useful since a network is then a collection of overlapping clubs and, so far, only a few papers have explored the Nash equilibrium with overlapping clubs structure. Bloch and Zenginobuz (2007) present a model of local public goods allowing spillovers between communities, and hence violating one of Tiebout's assumptions, which may be interpreted as a weighted network. Eshel, Samuelson, and Shaked (1998) and Corazzini and Gianazza (2008) adapt Ellison's (1993) local interaction model to public good games played on a spatial structure, which in a network setting correspond to a circulant network.

Of the policy questions that arise in connection with the private provision of public goods, the one of paramount importance is the effect of income redistribution. For a complete network, the question has, to a large extent, been settled by the neutrality result mentioned above. However, it appears that there has been no attempt in the economics of networks literature to explore whether the neutrality result holds beyond complete networks. To this effect, we provide an innovative approach based on the notion of main eigenvalue from spectral graph theory, due to Cvetković (1970), and on Bonacich centrality, first introduced to economics in the seminal paper of Ballester, Calvó-Armengol, and Zenou (2006), to show that the neutrality result will not, generally, hold beyond regular networks. We also expand on the links between main eigenvalues and Bonacich centrality to establish some results on the patterns of changes in the aggregate public good provision following income redistribution in

[5]Stiemke's Lemma, which is a strict version of Farkas–Minkowski's Lemma, has been a fundamental tool to characterize arbitrage-free portfolios in asset pricing theory.

some non-regular networks. Hence, we provide some useful predictions for the social planner or the network designer on which to base redistributive policies.

The paper is organized as follows. In Section 2, we present the model of private provision of public goods on networks. In Section 3, we establish the existence and uniqueness of a Nash equilibrium. In Section 4, we investigate the local stability of the Nash equilibrium. We explore the effect of income provision and introduce Bonacich centrality in Section 5 and we investigate the validity of the neutrality result in networks in Section 6. Section 7 provides some comparative statics results for the aggregate public good provision and Section 8 concludes the paper.

2. The model

There are n consumers embedded in a connected fixed network $\mathbf{g}$. Let $G = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumer i and consumer j are neighbors and $g_{ij} = 0$ otherwise. In particular, we assume that $g_{ii} = 0$ for each consumer $i = 1, \ldots, n$. Let $N_i = \{j \mid g_{ij} = 1\}$ denote the set of consumer i's neighbors. The adjacency matrix of the network, G, is symmetric with nonnegative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\max}(G) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = \lambda_{\min}(G)$, where $\lambda_{\max}(G)$ is the largest eigenvalue and $\lambda_{\min}(G)$ is the lowest eigenvalue of G. By the Perron–Frobenius Theorem, it holds that $\lambda_{\max}(G) \geq -\lambda_{\min}(G) > 0$ and, in particular, the equality $-\lambda_{\min}(G) = \lambda_{\max}(G)$ holds if and only if G is a bipartite network. Moreover, there is a matrix V such that $G = VDV^T$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal entries are the eigenvalues of G and V is a matrix whose columns, $v_1, v_2, \ldots, v_n$, are the corresponding eigenvectors of G that form an orthonormal basis of $\mathbb{R}^n$.

The preferences of each consumer $i = 1, \ldots, n$, are represented by the utility function $u_i(x_i, q_i + Q_{-i})$, where x_i is consumer i's private good consumption, q_i is consumer i's public good provision, and $Q_{-i} = \sum_{j \in N_i} q_j$ is the sum of public good provisions of consumer i's neighbors. For simplicity, we assume the public good can be produced from the private good with a unit-linear production technology. The utility function u_i is continuous, strictly increasing in both arguments, and strictly quasi-concave. Consumer i faces the following maximization problem:

$$\max_{x_i, q_i} \; u_i(x_i, q_i + Q_{-i})$$

$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where w_i is his income (exogenously fixed). It follows from the strict quasi-concavity that consumer i's public good provision is determined by a (single-valued) best-reply function f_i. At a Nash equilibrium $(q_1^*, q_2^*, \ldots, q_n^*)$, every consumer's choice is a best reply to the sum of his neighbors' public good provisions, that is, $q_i^* = f_i(Q_{-i}^*)$ for each consumer $i = 1, \ldots, n$.

Following a standard modification in the public goods literature, the utility maximization problem can be rewritten with consumer i choosing his (local) public good consumption, Q_i, rather than his public good provision, q_i, that is,

$$\max_{x_i, Q_i} u_i(x_i, Q_i)$$
$$\text{s.t.} \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i}.$$

If we ignore the last constraint $Q_i \geq Q_{-i}$ in the above maximization problem, we obtain a standard utility maximization problem of consumer demand theory. Hence a standard demand function for consumer i's public good consumption can be expressed by $Q_i = \gamma_i(w_i + Q_{-i})$, where $w_i + Q_{-i}$ may be interpreted as consumer i's "social income" and γ_i is the Engel curve for Q_i. In view of this, acknowledging the constraint $Q_i \geq Q_{-i}$ again leads to $Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\}$, which in turn implies

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\} = f_i(Q_{-i}). \tag{2.1}$$

Hence, consumers can only contribute a positive amount of the public good determined by their own demand for the public good, which in turn is a function of their (social) income and also their neighbors' public good provision.

3. Existence and uniqueness of the Nash equilibrium

In this section, we shall prove the existence and uniqueness of the Nash equilibrium for general networks and best-reply functions. In the case of a complete network, Bergstrom, Blume, and Varian (1986) rely on the assumption of normality of private and public goods to establish the existence and uniqueness of the Nash equilibrium. We introduce the following network-specific normality assumption.

Network normality. For each consumer $i = 1, \ldots, n$, the Engel curve γ_i is differentiable and it holds that $1 + \frac{1}{\lambda_{\min}(G)} < \gamma_i'(\cdot) < 1$.

The network normality assumption places bounds on the marginal propensity to consume the public good. Indeed, the left-hand-side inequality stipulates a strong normality of the public good, which depends on the lowest eigenvalue of the adjacency matrix G, while the right-hand-side inequality is the standard normality of the private good.

Theorem 3.1. *Assume network normality. Then there exists a unique Nash equilibrium in the private provision of public goods.*

Proof. The existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem. Suppose there are two Nash equilibria $q^1 = (q_1^1, q_2^1, \ldots, q_n^1) \neq (q_1^2, q_2^2, \ldots, q_n^2) = q^2$; then for each consumer $i = 1, \ldots, n$, it holds that

$$q_i^1 = f_i(Q_{-i}^1) = \max\{\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1, 0\}$$

and

$$q_i^2 = f_i(Q_{-i}^2) = \max\{\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2, 0\}.$$

Since $q^1 \neq q^2$ it follows that the set $C = \{i \mid Q_{-i}^1 \neq Q_{-i}^2\} \neq \emptyset$. Moreover, from the mean value theorem, for each consumer $i \in C$ there exists a real number β_i such that

$$\gamma_i(w_i + Q_{-i}^1) - \gamma_i(w_i + Q_{-i}^2) = \gamma_i'(\beta_i)(Q_{-i}^1 - Q_{-i}^2)$$

and hence

$$(\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1) - (\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2) = (1 - \gamma_i'(\beta_i))(Q_{-i}^2 - Q_{-i}^1).$$

Let $a = \max_{i \in C}\{1 - \gamma_i'(\beta_i)\}$; then it follows from the network normality assumption that for each consumer $i \in C$,

$$0 < 1 - \gamma_i'(\beta_i) \leq a < -\frac{1}{\lambda_{\min}(G)}.$$

For each consumer $i = 1, \ldots, n$, define s_i as follows:

$$s_i = \begin{cases} 1 & \text{if } Q_{-i}^1 \leq Q_{-i}^2, \\ -1 & \text{otherwise.} \end{cases}$$

Thus, for each consumer $i = 1, \ldots, n$, it holds that

$$\begin{aligned} 0 \leq s_i(q_i^1 - q_i^2) &= s_i(\max\{\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1, 0\} - \max\{\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2, 0\}) \\ &\leq s_i((\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1) - (\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2)) \\ &\leq s_i a(Q_{-i}^2 - Q_{-i}^1). \end{aligned}$$

Rearranging terms, since $q_i^1 \neq q_i^2$ at least for some i, it follows from the above inequalities that[6]

$$0 < (q^1 - q^2)(S, -(I + aG)S), \tag{3.1}$$

where I is the identity matrix and $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ is the diagonal matrix whose diagonal entries are s_i. The rest of the proof relies on a version of Stiemke's Lemma, as stated below.

[6]Consider $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$; then $x \geq 0$ if $x_i \geq 0$ for each $i = 1, \ldots, n$ and $x > 0$ if $x \geq 0$ and $x_i > 0$ for some i.

Stiemke's Lemma. *If A is an $m \times n$ real matrix, then one of the following mutually exclusive alternatives holds true:*

(1) There exists $x \in \mathbb{R}^n_{++}$ such that $Ax = 0$.

(2) There exists $y \in \mathbb{R}^m$ such that $y^T A > 0$.

Indeed, since inequality (3.1) implies alternative (2) holds for the matrix $(S, -(I + aG)S)$, it follows that there exists no $x \in \mathbb{R}^{2n}_{++}$ such that $(S, -(I + aG)S)x = 0$. That is, there exists no $x_1, x_2 \in \mathbb{R}^n_{++}$ with $(I + aG)Sx_2 = Sx_1$ which in turn implies that $(I + aG)S(\mathbb{R}^n_{++}) \cap S(\mathbb{R}^n_{++}) = \emptyset$. By continuity, it holds that $(I + aG)S(\mathbb{R}^n_{+}) \cap S(\mathbb{R}^n_{++}) = \emptyset$. From Minkowski's separating hyperplane theorem, there exists a hyperplane with normal $\pi \neq 0$, and a scalar α such that

(i) for all $u \in (I + aG)S(\mathbb{R}^n_{+})$, $\quad \pi \cdot u \leq \alpha$;
(ii) for all $v \in S(\mathbb{R}^n_{++})$, $\quad \pi \cdot v \geq \alpha$.

Since 0 belongs to the closure of the two sets, we can choose $\alpha = 0$. Moreover, it follows from (ii) in the separation theorem that $\pi \in S(\mathbb{R}^n_{+})$. Thus, it follows from (i) that $\pi^T(I + aG)\pi \leq 0$. Hence, $(I + aG)$ is not positive-definite, which is a contradiction. Therefore, there exists a unique Nash equilibrium.□

We have the following two corollaries:

Corollary 3.2. *(Bergstrom, Blume, and Varian (1986)) Assume that* **g** *is the complete network and that both private and public goods are normal goods. Then there exists a unique Nash equilibrium.*

Proof. When **g** is the complete network, it holds that $\lambda_{\min}(G) = -1$.[7] Thus, the normality of both private and public goods implies the network normality assumption and, hence, there exists a unique Nash equilibrium.□

[7] The adjacency matrix of the complete network is $J - I$, where J is the all-ones matrix. Since J has eigenvalues n and 0 with multiplicities 1 and $n - 1$, respectively, we see that the complete network has eigenvalues $n - 1$ and -1 with multiplicities 1 and $n - 1$.

Corollary 3.3. *(Bramoullé, Kranton, and D'Amours (2011)) Consider a linear strategic substitute game such that for each consumer* $i = 1, \ldots, n$*, it holds that* $q_i = \max\{1 - \alpha_i \sum_{j=1}^{n} g_{ij} q_j, 0\}$*, where* $\alpha_i \in]0, -\frac{1}{\lambda_{\min}(G)}[$*. Then there exists a unique Nash equilibrium.*

Proof. Observe that from (2.1), the linear strategic substitute game coincides with the public good game where for each consumer $i = 1, \ldots, n$, $\gamma_i'(\cdot) = 1 - \alpha_i$ and $w_i = \frac{1}{1-\alpha_i}$. Since $1 - \alpha_i \in]1 + \frac{1}{\lambda_{\min}(G)}, 1[$, it follows that the network normality assumption is satisfied and, hence, there exists a unique Nash equilibrium.□

4. Stability of the Nash equilibrium

We shall now investigate the issue of stability of the Nash equilibrium. Stability is of paramount importance to the study of comparative statics. If, following a small perturbation of parameters, the new equilibrium can be reached by a dynamic adjustment process, then the comparative statics analysis is strengthened. To explore the dynamic stability of the unique Nash equilibrium in the private provision of public goods, we consider a myopic adjustment process defined for each consumer $i = 1, \ldots, n$, by

$$\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} = \mu_i(f_i(Q_{-i}) - q_i),$$

where $\mu_1, \mu_2, \ldots, \mu_n > 0$ are the adjustment speeds (see Dixit (1986)).

Let $(q_1^*, q_2^*, \ldots, q_n^*)$ denote the unique Nash equilibrium. Before investigating stability, we partition the consumers into three sets: the set of active contributors

$$A = \{i \mid \gamma_i(w_i + Q_{-i}^*) > Q_{-i}^*\}$$

formed of consumers that would still contribute after a small perturbation of endowments; the set of knife-edge non-contributors

$$K = \{i \mid \gamma_i(w_i + Q_{-i}^*) = Q_{-i}^*\}$$

formed of consumers on the verge of becoming contributors; and the set of slack non-contributors

$$S = \{i \mid \gamma_i(w_i + Q_{-i}^*) < Q_{-i}^*\}$$

formed of consumers that would not contribute even after a small perturbation of endowments. The set of knife-edge non-contributors K is more likely to be empty, generically. Moreover, for notational simplicity, we also assume that $S = \emptyset$. Indeed, from the interlacing eigenvalue theorem, it holds that[8] $\lambda_{\min}(G) \le \lambda_{\min}(G \setminus S) < 0$ and, therefore, $0 < \frac{-1}{\lambda_{\min}(G)} \le \frac{-1}{\lambda_{\min}(G \setminus S)}$. Hence, if the network normality assumption holds for the network $\mathbf{g}$, it also holds for the network $\mathbf{g} \setminus S$.

The following result shows that the private provision Nash equilibrium is locally asymptotically stable under the same assumption required to ensure its uniqueness. Thus, local stability and uniqueness of equilibrium are closely related.

Theorem 4.1. *Assume network normality. Then the unique Nash equilibrium of the private provision of public goods is locally asymptotically stable.*

Proof. To study the local stability of the unique Nash equilibrium, we consider the Jacobian matrix at q^* :

$$J = -\begin{pmatrix} \mu_1 g_{11} & \mu_1 b_1 g_{12} & \dots & \mu_1 b_1 g_{1n} \\ \mu_2 b_2 g_{21} & \mu_2 g_{22} & \dots & \mu_2 b_2 g_{2n} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \mu_n b_n g_{n1} & \mu_n b_n g_{2n} & \dots & \mu_n g_{nn} \end{pmatrix},$$

where $b_i = 1 - \gamma_i'(w_i + Q^*_{-i})$. The unique Nash equilibrium is locally asymptotically stable if all eigenvalues of the Jacobian matrix J have negative real parts. Let $B = \text{diag}(b_1, b_2, \dots, b_n)$ and $U = \text{diag}(\mu_1, \mu_2, \dots, \mu_n)$; then it holds that $J = -U(I+BG)$. Let us consider the matrix $K = -J = U(I+BG)$. In the following lemma, we show that the eigenvalues of the matrix K are positive real numbers, which implies that the eigenvalues of the Jacobian matrix J are negative.

Lemma 4.2. *Assume network normality. Then the eigenvalues of the matrix K are positive real numbers.*

Proof. First, observe that the matrix

$$K = U(I + BG) = (UB)(B^{-1} + G)$$

[8]The matrix $G \setminus S$ is the adjacency matrix of the network $\mathbf{g} \setminus S$ obtained by deleting in the network $\mathbf{g}$ the nodes in S as well as the edges emanating from them.

is a symmetrizable matrix, as defined by Taussky (1968), since it is the product of the two symmetric matrices UB and $B^{-1}+G$, one of which (UB) is also positive-definite. Hence, the matrix K is similar to the symmetric matrix $U^{\frac{1}{2}}(I+B^{\frac{1}{2}}GB^{\frac{1}{2}})U^{\frac{1}{2}}$ since

$$K = U(I+BG) = (UB)^{\frac{1}{2}}[U^{\frac{1}{2}}(I+B^{\frac{1}{2}}GB^{\frac{1}{2}})U^{\frac{1}{2}}](UB)^{-\frac{1}{2}}.$$

Recall that the symmetric matrix $B^{\frac{1}{2}}GB^{\frac{1}{2}}$ has real eigenvalues. Moreover, it follows from Ostrowski (1959) that the eigenvalues of $B^{\frac{1}{2}}GB^{\frac{1}{2}}$ are given by $\theta_i\lambda_i$, where λ_i is an eigenvalue of G and θ_i lies between the smallest and the largest eigenvalues of B. From the network normality assumption, it follows that for each $i = 1, \dots, n$,

$$0 < \min_i\{1-\gamma_i'(w_i+Q_{-i}^*)\} \le \theta_i \le \max_i\{1-\gamma_i'(w_i+Q_{-i}^*)\} < -\frac{1}{\lambda_{\min}(G)}.$$

Consequently, the eigenvalues of $I + B^{\frac{1}{2}}GB^{\frac{1}{2}}$, given by $1+\theta_i\lambda_i$, are positive since for each $i = 1, \dots, n$, it holds that

$$0 = 1 - 1 < 1 + \theta_i\lambda_{\min}(G) \le 1 + \theta_i\lambda_i.$$

From Ostrowski (1959) again, it follows that the eigenvalues of $U^{\frac{1}{2}}(I+B^{\frac{1}{2}}GB^{\frac{1}{2}})U^{\frac{1}{2}}$ are given by $\nu_i(1+\theta_i\lambda_i)$, where $0 < \min_i\mu_i \le \nu_i \le \max_i\mu_i$, which, therefore, implies that $0 < \nu_i(1+\theta_i\lambda_i)$. Since K is similar to $U^{\frac{1}{2}}(I+B^{\frac{1}{2}}GB^{\frac{1}{2}})U^{\frac{1}{2}}$, it follows that the eigenvalues of K are also positive.□

5. Income redistribution and Bonacich centrality

Ballester, Calvó-Armengol, and Zenou (2006) show that in the case of linear best-reply functions the Nash equilibrium actions are proportional to the Bonacich centrality vector. Bonacich centrality, due to Bonacich (1987), is defined for $a < \frac{1}{\lambda_{\max}(G)}$ by the vector

$$\mathbf{b}(G,a) = \mathbf{1}^T(I-aG)^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{1}^T G^k,$$

where $\mathbf{1}$ is the all-ones vector. Since the i^{th} entry of the vector $\mathbf{1}^T G^k$ denotes the number of walks of length k in G terminating at i, it follows that the i^{th} entry $\mathbf{b}_i(G,a)$ of the Bonacich centrality vector is the sum of all walks in G terminating at i weighted by a to the power of their length. In that sense, Bonacich centrality is intepreted as a measure of prestige, power, and network influence

In the following, in the case of nonlinear best-reply functions we show that equilibrium actions and Bonacich centrality are also closely related. More specifically, we establish that the effect of income redistribution on the aggregate public good provision may be determined by a generalization of Bonacich centrality. Let $t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{R}^n$, where t_i denotes the income transfer made to consumer i. The income transfer may be either a tax ($t_i < 0$) or a subsidy ($t_i \geq 0$). The social planner or network designer is constrained to balance his budget; hence $\mathbf{1} \cdot t = \sum_{i=1}^{n} t_i = 0$. Let $(q_1^*, q_2^*, \ldots, q_n^*)$ (resp. $(q_1^t, q_2^t, \ldots, q_n^t)$) denote the unique Nash equilibrium before income redistribution (resp. after income redistribution) and $Q^* = \sum_i q_i^*$ (resp. $Q^t = \sum_i q_i^t$) denote the aggregate public good provision before income redistribution (resp. after income redistribution). Similar to Bergstrom, Blume, and Varian (1986), we choose t relatively small in magnitude so that the set of active contributors remains unchanged after income redistribution. For simplicity, as in our stability analysis, we also assume that all consumers are active contributors. Hence, it follows that for each consumer $i = 1, 2, \ldots, n$,

$$q_i^t - q_i^* = (\gamma_i(w_i + t_i + Q_{-i}^t) - Q_{-i}^t) - (\gamma_i(w_i + Q_{-i}^*) - Q_{-i}^*).$$

From the mean value theorem it follows that for each i such that $t_i + Q_{-i}^t \neq Q_{-i}^*$, there exists a real number β_i such that

$$q_i^t - q_i^* = \gamma_i'(\beta_i)(t_i + Q_{-i}^t - Q_{-i}^*) - (Q_{-i}^t - Q_{-i}^*). \tag{5.1}$$

Define a_i as follows:

$$a_i = \begin{cases} 1 - \gamma_i'(\beta_i) & \text{if } t_i + Q_{-i}^t \neq Q_{-i}^*, \\ 1 - \gamma_i'(Q_{-i}^*) & \text{otherwise,} \end{cases}$$

and let us consider the diagonal matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$.

Proposition 5.1. *Assume network normality. Then it holds that*

$$q^t - q^* = (I + AG)^{-1}(I - A)t.$$

Proof. First, rearranging terms in (5.1), it follows that for each i such that $t_i + Q_{-i}^t \neq Q_{-i}^*$, it holds that

$$q_i^t - q_i^* + a_i \sum_{j \in N_i} (q_j^t - q_j^*) = (1 - a_i)t_i. \tag{5.2}$$

Moreover, observe that (5.2) also holds trivially for each i such that $t_i + Q_{-i}^t = Q_{-i}^*$. Consequently, it holds that $(I + AG)(q^t - q^*) = (I - A)t$. Applying Lemma 4.2 for $B = A$ and $U = I$, it follows that the eigenvalues of the matrix $I + AG$ are positive, which implies that $I + AG$ is invertible. Hence, $q^t - q^* = (I + AG)^{-1}(I - A)t$.□

Proposition 5.1 is useful for the comparative statics analysis of the private provision of public goods since one can relate the changes in each consumer's public good provision to the income redistribution t and the marginal propensities a_i. In particular, if one is concerned with the change in the aggregate public good provision, it follows that

$$Q^t - Q^* = \mathbf{1} \cdot (q^t - q^*) = \mathbf{b}^{dw}(G, -A)(I - A)t, \tag{5.3}$$

where

$$\mathbf{b}^{dw}(G, -A) = \mathbf{1}^T (I + AG)^{-1}.$$

The vector $\mathbf{b}^{dw}(G, -A)$, which is well defined, may be thought of as a "diagonally weighted" Bonacich centrality where each node carries a different weight. The "diagonally weighted" Bonacich centrality summarizes information concerning each node's impact on the aggregate public good provision. Recent contributions of Candogan, Bimpikis, and Ozdaglar (2010) and Golub and Carlos (2010) have proposed other useful generalizations of Bonacich centrality, which, provided that they are well defined, characterize equilibria outcomes in some classes of games.

6. Neutrality in networks

In this section, we shall explore the effect of income redistribution on the aggregate public good provision. For a complete network, the invariance result of Warr (1983) and Bergstrom, Blume, and Varian (1986), the so-called neutrality result, shows that income redistributions that preserve the set of contributors will have no effect on the aggregate public good provision or individual private good consumption. The following proposition provides a proof of the neutrality result based on network analysis of the private provision of public goods.

Proposition 6.1. *Assume network normality and that* $\mathbf{g}$ *is the complete network.*

Then it holds that $q^t - q^* = t$.

Proof. First, observe that, from the network normality assumption, it follows that both matrices $I + AG$ and $I - A$ are invertible. Moreover, it holds that

$$(I - A)^{-1}(I + AG) = \begin{pmatrix} 1 + \frac{a_1}{1-a_1} & \frac{a_1}{1-a_1} & \cdots & \frac{a_1}{1-a_1} \\ \frac{a_2}{1-a_2} & 1 + \frac{a_2}{1-a_2} & \cdots & \frac{a_2}{1-a_2} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \frac{a_n}{1-a_n} & \frac{a_n}{1-a_n} & \cdots & 1 + \frac{a_n}{1-a_n} \end{pmatrix}.$$

Let $u = (\frac{a_1}{1-a_1}, \frac{a_2}{1-a_2}, \ldots, \frac{a_n}{1-a_n})^T$; then it holds that $(I-A)^{-1}(I+AG) = I + u\mathbf{1}^T$. From the Sherman–Morrison formula (see, for example, Maddala (1977, p. 446)), it follows that

$$(I+AG)^{-1}(I-A) = (I+u\mathbf{1}^T)^{-1} = I - \frac{1}{1+\sum_{i=1}^n u_i} u\mathbf{1}^T.$$

Hence, it follows from Proposition 5.1 that

$$q^t - q^* = (I - \frac{1}{1+\sum_{i=1}^n u_i} u\mathbf{1}^T)t = t.\Box$$

What is remarkable in the neutrality result is that, regardless of the form of the preferences, each consumer adjusts his public good provision by precisely the amount of the income transfer made to him, provided that the set of contributors remains unchanged.

Remark 1. An alternative way to establish the invariance of the aggregate public good provision that avoids calculating the inverse of the matrix $(I-A)^{-1}(I+AG)$ is to notice that the matrix has constant column sums. This implies that $\mathbf{1}^T$ is a left eigenvector for the matrix $(I-A)^{-1}(I+AG)$ and it holds that

$$Q^t - Q^* = \mathbf{b}^{dw}(G, -A)(I-A)t = \mathbf{1}^T(I+AG)^{-1}(I-A)t = \frac{1}{1+\sum_{i=1}^n \frac{a_i}{1-a_i}} \mathbf{1} \cdot t = 0.$$

We now turn our attention to investigate neutrality of income redistribution in general networks. We will focus only on the first part of the invariance result, that is, whether the aggregate public good provision is independent of income redistribution. In principle, provided that it holds, the neutrality result is not special to a particular form of preferences. Therefore, we can also focus our analysis on preferences yielding parallel affine Engel curves, the so-called Gorman polar form, of which the Cobb-Douglas preferences are a special case.[9]

[9]This also corresponds to the class of games studied by Bramoullé, Kranton, and D'Amours (2011) where all consumers have the same linear best-reply function. In the literature, the theoretical and empirical attraction of preferences of the Gorman polar form is that one can treat a society of utility-maximizing individuals as a single consumer. Such a concept, albeit different, bears a great methodological similarity to the concept of potential games of Monderer and Shapley (1996).

In the following, we will introduce the concept of main eigenvalue, due to Cvetković (1970), from spectral graph theory, to pursue our analysis of the effect of income redistribution on the aggregate public good provision. An eigenvalue μ_i of the adjacency matrix G is called a main eigenvalue if it has a (unit) eigenvector u_i not orthogonal to $\mathbf{1}$, that is, $\mathbf{1} \cdot u_i \neq 0$. Since for eigenvalues with multiplicity greater than one we can choose the corresponding eigenvectors in such a way that, at most, one of them is not orthogonal to $\mathbf{1}$, without loss of generality, we may also assume that $u_i \in \{v_1, v_2, \ldots, v_n\}$, the orthonormal basis of $\mathbb{R}^n$ formed by the eigenvectors of G. In addition, it also holds that the main eigenvalues of G are distinct and may, consequently, be ordered $\mu_1 > \mu_2 > \ldots > \mu_s$. Recall that, by the Perron–Frobenius Theorem, the principal eigenvector v_1 has positive entries and, hence, $\mu_1 = \lambda_{\max}(G)$. The set of main eigenvalues $M = \{\mu_1, \mu_2, \ldots, \mu_s\}$ is called the main part of the spectrum. Cvetković (1970) shows that the number of walks in a network is closely related to the main part of the spectrum. Indeed, let $W_k = \mathbf{1}^T G^k \mathbf{1}$ denote the number of walks of length k in G; then there exist constants $c_1, c_{2,} \ldots, c_s$ such that for every k, $W_k = \sum_{i=1}^{s} c_i \mu_i^k$. The following result provides an easy characterization of the main part of the spectrum.

Theorem 6.1. *(Harary and Schwenk (1979)) The following statements are equivalent for a network* $\mathbf{g}$.*:*

(i) M is the main part of the spectrum.

(ii) M is the minimum set of eigenvalues the span of whose eigenvectors includes $\mathbf{1}$.

(iii) M is the set of those eigenvalues which have an eigenvector not orthogonal to $\mathbf{1}$.

The following theorem, based on the concept of main eigenvalues, shows that the neutrality result of Warr (1983) and Bergstrom, Blume, and Varian (1986) has a limited scope of application beyond regular networks.

Theorem 6.2. *Assume network normality and that the preferences of consumers yield parallel affine Engel curves, that is, $\gamma_i'(\cdot) = 1 - a$ for each consumer $i = 1, \ldots, n$. Then the aggregate public good provision is invariant to income redistribution if and only if the network is regular.*

Proof. From the network normality assumption, it follows that the matrix $I + aG$ has positive eigenvalues and so is invertible. Since $G = VDV^T$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ whose diagonal entries are the eigenvalues of G and V is a matrix whose columns, $v_1, v_2, \ldots, v_n$, are the corresponding eigenvectors of G that form an orthonormal basis of $\mathbb{R}^n$, it holds that

$$(I + aG)^{-1} = V(I + aD)^{-1}V^T = \sum_{i=1}^{n} \frac{1}{1 + a\lambda_i} v_i v_i^T.$$

Moreover, since $\{u_1, u_2, \ldots, u_s\} \subset \{v_1, v_2, \ldots, v_n\}$, it follows that

$$\mathbf{b}(G, -a) = \mathbf{1}^T(I + aG)^{-1} = \sum_{i=1}^{n} \frac{\mathbf{1} \cdot v_i}{1 + a\lambda_i} v_i^T = \sum_{i=1}^{s} \frac{\mathbf{1} \cdot u_i}{1 + a\mu_i} u_i^T. \tag{6.1}$$

From (5.3), it follows that $Q^t - Q = \mathbf{1} \cdot (q^t - q) = (I - a)\mathbf{b}(G, -a)t$, and, hence, the aggregate public good provision is invariant to income redistribution if and only if there exists a real number λ such that $\mathbf{b}(G, -a) = \lambda \mathbf{1}^T$, which from (6.1) is equivalent to

$$\mathbf{b}(G, -a) = \lambda \mathbf{1}^T = \lambda \sum_{i=1}^{s} \mathbf{1} \cdot u_i u_i^T = \sum_{i=1}^{s} \frac{\mathbf{1} \cdot u_i}{1 + a\mu_i} u_i^T. \tag{6.2}$$

Recall that the main eigenvectors $u_1, u_2, \ldots, u_s$ are linearly independent. Thus, (6.2) is equivalent to

$$\lambda = \frac{1}{1 + a\mu_1} = \frac{1}{1 + a\mu_2} = \ldots = \frac{1}{1 + a\mu_s},$$

which, since the main eigenvalues $\mu_1, \mu_2, \ldots, \mu_s$ are distinct, holds if and only if $s = 1$. From (2) in Theorem 6.1, it follows that $s = 1$ if and only if $\mathbf{1}$ is an eigenvector of G, which is equivalent to $\mathbf{g}$ being a regular network.□

The above result shows that neutrality fails to hold in non-regular networks since the aggregate provision is affected by income redistribution. It is worth noting that even in regular but not complete networks, neutrality holds only for the aggregate public good provision and it may be easily observed that either the private good consumption or the public good consumption may have been changed for some consumers.

Remark 2. To the best of our knowledge, the equality $\mathbf{b}(G, -a) = \sum_{i=1}^{s} \frac{\mathbf{1} \cdot u_i}{1+a\mu_i} u_i^T$ in (6.1) is the first formulation of Bonacich centrality in terms of the main part of the spectrum. Note that the non-main eigenvalues do not contribute to Bonacich centrality since the corresponding eigenvectors are orthogonal to $\mathbf{1}$.

Remark 3. Often, each eigenvector v_i of G may determine a measure of relative importance in the network where the weight of a particular node j corresponds to the j^{th} entry of the eigenvector v_i normalized by the sum of the entries of the various nodes. Such a measure is self-referential since, by the definition of an eigenvector, the weight of a node is proportional to the sum of the weights of its neighbors. Consequently, it is not possible for a non-main eigenvalue $v_i \neq u_1, u_2, \ldots, u_s$ to generate a measure of relative importance since the sum $\sum_{j=1}^{n} v_i^j = \mathbf{1} \cdot v_i = 0$ and the entries of the various nodes eventually cancel each other out.

7. Comparative statics

In view of the limited redistributive neutrality in general networks, it may be desirable for the social planner or network designer to learn about the pattern of changes in aggregate public good provision following income redistribution. From a purely welfare standpoint, it is worth noting that, in spite of the typical suboptimality of the Nash equilibrium in the private provision of public goods, an increased aggregate public good provision in another equilibrium, achieved after income redistribution, may not necessarily support a Pareto improvement. Setting aside the questions of (second-best) optimality, one may argue that the aggregate public good provision may serve as a benchmark for free-riding or aggregate activity in the network, or may affect the social welfare function of the social planner or network designer separately.

7.1. Networks with exactly two main eigenvalues ($s = 2$).

This is the first instance of non-regular networks in which the neutrality result fails to hold. The simplest examples of networks with just two main eigenvalues are the complete bipartite networks and the networks obtained from deleting a node in a strongly regular network. More generally, it holds that a network $\mathbf{g}$ and its complement network $\overline{\mathbf{g}}$ have the same number of main eigenvalues. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)^T$ denote the vector of degree centrality, where d_i is the degree of node i, so that $\mathbf{d} = G\mathbf{1}$.

Proposition 7.1. *Assume network normality and that the preferences of consumers yield parallel affine Engel curves, that is,* $\gamma_i'(\cdot) = 1 - a$ *for each consumer* $i = 1, \ldots, n$. *If network* $\mathbf{g}$ *has exactly two main eigenvalues, then it holds that*

$$Q^t - Q^* = \frac{-a(1-a)}{(1+a\mu_1)(1+a\mu_2)}\mathbf{d} \cdot t.$$

Proof. From Hagos (2002), it follows that if μ_1 and μ_2 are the two main eigenvalues of G, then the associated unit eigenvectors are, respectively,

$$u_1 = \frac{(G - \mu_2 I)\mathbf{1}}{\sqrt{(\mu_1 - \mu_2)\mathbf{1}^T(G - \mu_2 I)\mathbf{1}}} \text{ and } u_2 = \frac{(G - \mu_1 I)\mathbf{1}}{\sqrt{(\mu_2 - \mu_1)\mathbf{1}^T(G - \mu_1 I)\mathbf{1}}}.$$

Hence, it follows from (5.3) and (6.1) that

$$\begin{aligned}
Q^t - Q^* &= (1-a)\left(\frac{\mathbf{1} \cdot u_1}{1+a\mu_1}u_1 + \frac{\mathbf{1} \cdot u_2}{1+a\mu_2}u_2\right) \cdot t \\
&= \frac{1-a}{(1+a\mu_1)(1+a\mu_2)}[(1+a\mu_2)(\mathbf{1} \cdot u_1)u_1 + (1+a\mu_1)(\mathbf{1} \cdot u_2)u_2] \cdot t \\
&= \frac{1-a}{(1+a\mu_1)(1+a\mu_2)}[(1+a\mu_2)\frac{(\mathbf{1}^T(G-\mu_2 I)\mathbf{1})(G-\mu_2 I)\mathbf{1}}{(\sqrt{(\mu_1-\mu_2)\mathbf{1}^T(G-\mu_2 I)\mathbf{1}})^2} \\
&+ (1+a\mu_1)\frac{(\mathbf{1}^T(G-\mu_1 I)\mathbf{1})(G-\mu_1 I)\mathbf{1}}{(\sqrt{(\mu_2-\mu_1)\mathbf{1}^T(G-\mu_1 I)\mathbf{1}})^2}] \cdot t \\
&= \frac{1-a}{(1+a\mu_1)(1+a\mu_2)}[\frac{(1+a\mu_2)(G-\mu_2 I)\mathbf{1} - (1+a\mu_1)(G-\mu_1 I)\mathbf{1}}{\mu_1 - \mu_2}] \cdot t \\
&= \frac{1-a}{(1+a\mu_1)(1+a\mu_2)}[\frac{a(\mu_2-\mu_1)G\mathbf{1}}{\mu_1-\mu_2}] \cdot t = \frac{-a(1-a)}{(1+a\mu_1)(1+a\mu_2)}\mathbf{d} \cdot t. \square
\end{aligned}$$

Hence, for networks with, at most, two main eigenvalues ($s = 1, 2$), our results generate precise and clear predictions about the effect of income redistribution on the aggregate public good provision. Furthermore, it turns out that the aggregate public good provision is determined by the degree centrality rather than the more sophisticated Bonacich centrality.[10] Our results are in line with similar observations in the economics of networks literature. König, Tessone, and Zenou (2009) present a model

[10] It is worth noting that degree centrality is a local network measure since only walks of length 1 are considered, unlike Bonacich centrality, which is a global network measure since all walks are

of dynamic network formation where the degree and Bonacich centrality rankings coincide and Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) emphasize the importance of the degree centrality as a measure of immediate influence and local knowledge of the network.

7.2. **Asymptotic behavior of Bonacich centrality.** It is well known that the Bonacich centrality becomes asymptotic to the eigenvector centrality, defined by the principal eigenvector v_1, when the attenuation factor of the Bonacich centrality approaches $\frac{1}{\lambda_{\max}(G)}$ from below. In our case, since we deal with the private provision of public goods which belongs to the general class of games of strategic substitutes, the attenuation factor of the Bonacich centrality is negative and, thus, the behavior of Bonacich centrality must be explored at the other end of the spectrum, that is, in the neighborhood of $\frac{1}{\lambda_{\min}(G)}$.

In the following, we show that the concept of main eigenvalue is also relevant for the study of the asymptotic behavior of Bonacich centrality. Indeed, our analysis of preferences that have parallel affine Engel curves reveals that the key issue is whether the lowest eigenvalue is a main eigenvalue.

Proposition 7.2. *If $\lambda_{\min}(G) = \mu_s$, then, when a approaches $\frac{-1}{\lambda_{\min}(G)}$ from below, the Bonacich centrality $\mathbf{b}(G, -a)$ is asymptotic to the eigenvector u_s.*

Proof. First, observe that from (6.1) it holds that

$$\begin{aligned}
\lim_{a\uparrow\frac{-1}{\lambda_{\min}(G)}} \left(\frac{1 + a\lambda_{\min(G)}}{\mathbf{1}\cdot u_s}\right)\mathbf{b}(G, -a) &= \lim_{a\uparrow\frac{-1}{\lambda_{\min}(G)}} \left(\frac{1 + a\lambda_{\min(G)}}{\mathbf{1}\cdot u_s}\right) \sum_{i=1}^{s} \frac{\mathbf{1}\cdot u_i}{1 + a\mu_i} u_i^T \\
&= \lim_{a\uparrow\frac{-1}{\lambda_{\min}(G)}} \sum_{i=1}^{s-1} \frac{(1 + a\lambda_{\min(G)})(\mathbf{1}\cdot u_i)}{(\mathbf{1}\cdot u_s)(1 + a\mu_i)} u_i^T + u_s^T \\
&= u_s^T. \square
\end{aligned}$$

Proposition 7.2 may have a useful policy implication for non-regular networks with $\lambda_{\min}(G) = \mu_s$. Indeed, in the case of general preferences having Engel curves close enough to an affine curve with a slope of $1+\frac{1}{\lambda_{\min}(G)}$, it holds by a continuity argument that the "diagonally weighted" Bonacich centrality $\mathbf{b}^{dw}(G, -A)$ is also asymptotic

considered. Moreover, observe that for networks with, at most, two main eigenvalues, neutrality still holds for income redistributions amongst consumers with the same degree.

to the eigenvector u_s. Furthermore, since the eigenvector u_s is orthogonal to the positive entries principal eigenvector u_1, it follows that the eigenvector u_s has both positive and negative entries. Thus, income redistributions that follow the sign patterns of the eigenvector u_s will have a predictable impact on the aggregate public good provision. Finally, for non-regular networks with $\lambda_{\min}(G) = \mu_s$, the predictions are less clear and vary according to the vector of Bonacich centrality

$$\mathbf{b}(G, \frac{1}{\lambda_{\min}(G)}) = \lambda_{\min}(G) \sum_{i=1}^{s} \frac{\mathbf{1} \cdot u_i}{\lambda_{\min}(G) - \mu_i} u_i^T.$$

8. Conclusion

In this paper, we have established that beyond regular networks, consumers are no longer able to offset income transfers by changes in their public good provisions. Our result restores, to some extent, the role of income redistribution and tax-financed government contribution as main channels for policy intervention. In the literature, various lines of research have been proposed to counter the paradigm of neutrality of income redistribution. Often, the reason neutrality break down appears to hinge on the imperfect substitution amongst the various consumers' provisions (see, for example, Andreoni (1990)). Our result suggests that neutrality fails in the private provision of public goods on non-regular networks for similar reasons. However, unlike the various behavioral and technological explanations in the literature, the lack of perfect substitution seems to be brought about by the inherent degree heterogeneity of non-regular networks.

Finally, most of our results, including existence, uniqueness, and stability of the Nash equilibrium in the private provision of public goods on networks, are based on properties of the best-reply functions and, hence, may accommodate the general class of games of strategic substitutes on networks with nonlinear best-reply functions, which are the cornerstone in the study of various areas of economics (see Bulow, Geanakoplos, and Klemperer (1985) and subsequent literature).

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